Fair Bankruptcy Solutions Under Interval Uncertainty

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1. What is a bankruptcy problem

- A company goes bankrupt if the total amount of its assets is smaller than the total amount of debts.
- Some of the debts have priority.
- E.g., according to the US labor law, salary needs to be paid in full, irrespective of debts to others.
- Once these priority debts are paid, we face a problem of:
  - how to divide the remaining assets A
  - between the creditors to whom the company owes amounts

$$d_1, \ldots, d_n.$$

2. How this problem is usually solved

- A usual solution is to make payments proportional to debts, i.e., depending on the ratio between the assets and the debts,
  - 10 cents per dollar,
  - 50 cents per dollar, etc.
- In general, the amount g_i given to the i-th creditor is equal to

$$g_i = d_i \cdot \frac{A}{\sum_{j=1}^{n} d_j}.$$

3. Need to take interval uncertainty into account

- In some cases, the debt is purely monetary, and its amount d_i is known exactly.
- In many practical situations, however, the situation is more complicated.
- For many creditors, we only know the bounds $\underline{d}_i \le d_i \le \overline{d}_i$ of the actual debt amount.
- How should we divide the assets in this situation?

4. Case of interval uncertainty: how is this problem solved now

- Several papers describe how to solve the bankruptcy problem under interval uncertainty.
- For example, Branzei at al. (2004) suggest:
  - selecting a single value d_i within each interval, and
  - then using these values d_i to divide the assets.
- For example, to select d_i, we can use Hurwicz optimism-pessimism criterion:
  - we agree on some value $\alpha \in [0, 1]$ and
  - we take $d_i = \alpha \cdot \overline{d}_i + (1 - \alpha) \cdot \underline{d}_i$.
- A more complex scheme was proposed in Li et al. (2021).
- In this talk, we show that a natural formalization of fairness uniquely determines Hurwicz-based solutions.
- These solutions are thus recommended as the fair ones.

5. How to describe fairness

- Fairness means, first, that:
  - if the debt d_i to creditor i is smaller than or equal to the debt d_j to creditor j,
  - then the payment g_i to creditor i should be smaller than or equal to the payment to creditor j.
- Second, fairness means that two creditors should not gain or lose by joining together; in other words:
  - if for debts $d_1, d_2, d_3, \ldots, d_n$, we had payments $g_1, g_2, g_3, \ldots, g_n$,
  - them for debts $d_1 + d_2, d_3, \ldots, d_n$, we should have payments

$$g_1 + g_2, g_3, \ldots, g_n.$$

- It also makes sense to require that:
  - if in two situations, debts are close,
  - then payments should be close.
- I.e., payments should be a continuous function of debts.

6. What if we impose fairness requirements in situations when we know the exact amount of debts

- Our main objective is to consider the case of interval uncertainty.
- To do it, let us first analyze what happens when we know the exact amount of debt.
- Let $A < D$ be two positive numbers.
- We will call A the *amount of assets*, and we will call D *the amount of debt*.
- By a *solution* to the bankruptcy problem (or simply *solution*, for short), we mean a function S that:
  - maps every tuple $\langle d_1, \ldots, d_n \rangle$ of positive real numbers for which

$$d_1 + \ldots + d_n = D$$

  - into a tuple of non-negative real numbers $\langle g_1, \ldots, g_n \rangle$ for which

$$g_1 + \ldots + g_n = A.$$

7. What if we know the exact amount of debts

- We say that the solution $S(\langle d_1, \ldots, d_n\rangle) = \langle g_1, \ldots, g_n\rangle$ is *fair* if it satisfies the following two requirements for each tuple d_i:
  - if $d_i \le d_j$, then $g_i \le g_j$;
  - we have $S(\langle d_1 + d_2, d_3, \ldots, d_n\rangle) = \langle g_1 + g_2, g_3, \ldots, g_n\rangle$.
- We say that the solution S is *continuous*, if for every n:
  - if $d_i^{(k)} \to d_i$ for all i, $S(\langle d_1^{(k)}, \ldots, d_n^{(k)}\rangle) = \langle g_1^{(k)}, \ldots, g_n^{(k)}\rangle$, and $g_i^{(k)} \to g_i$ for all i,
  - then $S(\langle d_1, \ldots, d_n, A\rangle) = \langle g_1, \ldots, g_n\rangle$.

8. Case when we know the debts: result

- **Proposition.** *For each solution S, the following two conditions are equivalent to each other:*
  - *the solution is fair and continuous,*
  - *the solution has the form $g_i = d_i \cdot (A/D)$.*
- So, the usual solution is the only one which is fair (and continuous).
- **Proof.** It is easy to check that the above solution is fair and continuous.
- So, to complete the proof, it is sufficient to prove that every fair continuous solution S has this form.
- Indeed, let S be a fair and continuous solution.
- For every natural number N, we can consider the tuple $\langle d_1, \ldots, d_N \rangle = \langle D/N, \ldots, D/N \rangle$ consisting of N equal debt values.
- By the first fairness requirement, since the debts d_i are all equal, the payments g_i are also all equal.

9. Proof (cont-d)

- Since $g_1 + \ldots + g_N = A$, this means that $N \cdot g_i = A$ hence $g_i = A/N$, and the payments tuple has the form $\langle g_1, \ldots, g_N \rangle = \langle A/N, \ldots, A/N \rangle$.
- Let us consider a sequence of natural numbers $k_1, \ldots, k_n$ for which $k_1 + \ldots + k_n = N$.
- Then, the tuple $\langle k_1 \cdot (D/N), \ldots, k_n \cdot (D/N) \rangle$ can be obtained from the tuple $\langle 1/N, \ldots, 1/N \rangle$ by:
  - adding up the first k_1 terms,
  - then the next k_2 terms, etc.
- So, due to the second fairness requirements, the resulting payment tuple $\langle g_1, \ldots, g_n \rangle$ can be obtained from the tuple $\langle A/N, \ldots, A/N \rangle$:
  - by adding the first k_1 terms,
  - then the next k_2 terms, etc.
- Thus, the payment tuple has the form $\langle k_1 \cdot (A/N), \ldots, k_n \cdot (A/N) \rangle$.

10. Proof (cont-d)

- In other words, for each debt $d_i = k_i \cdot (D/N)$, the payment is equal to $g_i = k_i \cdot (A/N)$.
- From $d_i = k_i \cdot (D/N)$, we conclude that $k_i = d_i \cdot (N/D)$, hence

$$g_i = k_i \cdot (A/N) = d_i \cdot (N/D) \cdot (A/N) = d_i \cdot (A/D).$$

- So indeed $g_i = d_i \cdot (A/D)$.
- We have proved the desired equality for all the cases when:
  - for all the debts d_i,
  - we have $d_i = k_i \cdot (D/N)$ for some integer k_i, i.e., when

$$d_i/D = k_i/N.$$

11. Proof (cont-d)

- Any real number d_i/D can be approximated – with accuracy $1/N$ – by an appropriate fraction k_i/N.
- As N increases, the fraction tends to d_i/D.
- Thus, since the solution S is continuous, in the limit, we will have the desired equality for all possible real values d_i.
- The proposition is proven.

12. Comment

- At first glance, it may sound reasonable to also require that:
  - if we combine two bankruptcy problems together,
  - then in the combined problem, each creditors should receive the sum of what he/she would receive in each solutions.
- In other words:
  - if we have $S(\langle d_1, \ldots, d_n\rangle) = \langle g_1, \ldots, g_n\rangle$ and
  $$S(\langle d'_1, \ldots, d'_n\rangle) = \langle g'_1, \ldots, g'_n\rangle,$$
  - then we should have $S(\langle d_1+d'_1, \ldots, d_n+d'_n\rangle) = \langle g_1+g'_1, \ldots, g_n+g'_n\rangle$.
- This requirement is explicitly mentioned in Li et al. (2021).
- Let us show, however, that the fair solution does not have this property.

13. Comment (cont-d)

- Let us take $d_1 = 4$, $d_2 = 1$, and $A = 2$.
- Then $D = d_1 + d_2 = 4 + 1 = 5$, so $A/D = 2/5 = 0.4$, $g_1 = d_1 \cdot (A/D) = 4 \cdot 0.4 = 1.6$, and $g_2 = d_2 \cdot (A/D) = 1 \cdot 0.4 = 0.4$;
- Let us also take $d'_1 = d'_2 = 1$ and $A' = 1$.
- Then $D' = d'_1 + d'_2 = 1 + 1 = 2$, so $A'/D' = 1/2 = 0.5$, and $g'_i = d'_i \cdot (A'/D') = 1 \cdot 0.5 = 0.5$.
- On the other hand, for $d_1 + d'_1 = 5$, $d_2 + d'_2 = 2$, and $A + A' = 3$, we have $D + D' = 7$, so $(A + A')/(D + D') = 3/7$.
- Thus, for the first creditor, the payment is

$$(d_1 + d'_1) \cdot ((A + A')/(D + D')) = 5 \cdot (3/7) = 15/7 = 2 + 1/7.$$

- This is different from this creditor's summary payment $g_1 + g'_1 = 1.6 + 0.5 = 2.1$ in two original situations.

14. Interval sum and interval order: reminder

- In the case of interval uncertainty,
- if we only know that the debt d_1 is in the interval $[\underline{d}_1, \overline{d}_1]$ and that the debt d_2 is in the interval $[\underline{d}_2, \overline{d}_2]$,
- then the only conclusion we can make about the summary debt $d_1 + d_2$ to these two creditors is that this sum belongs to the interval

$$[\underline{d}_1 + \underline{d}_2, \overline{d}_1 + \overline{d}_2].$$

- This interval is known as the *sum* $[\underline{d}_1, \overline{d}_1] + [\underline{d}_2, \overline{d}_2]$ of the two intervals $[\underline{d}_1, \overline{d}_1]$ and $[\underline{d}_2, \overline{d}_2]$.
- A natural order is component-wise.
- We say that the debt $[\underline{d}_i, \overline{d}_i]$ to creditor i is smaller than or equal to the debt $[\underline{d}_j, \overline{d}_j]$ to creditor j if $\underline{d}_i \le \underline{d}_j$ and $\overline{d}_i \le \overline{d}_j$.

15. Interval case: definitions

- Let A be a positive real numbers and let $[\underline{D}, \overline{D}]$ be an interval for which $0 < \underline{D}$ and $A < \overline{D}$.
- We will call A *the amount of assets*, and we will call $[\underline{D}, \overline{D}]$ *the amount of debt.*
- By a *solution* to the bankruptcy problem (or simply *solution*, for short), we mean a function S that:
  - maps every tuple $\langle [\underline{d}_1, \overline{d}_1] \ldots, [\underline{d}_n, \overline{d}_n] \rangle$ of intervals for which $0 \le \underline{d}_i$, numbers for which $\underline{d}_1 + \ldots + \underline{d}_n = \underline{D}$, and $\overline{d}_1 + \ldots + \overline{d}_n = \overline{D}$
  - into the same-size tuple of non-negative real numbers $\langle g_1, \ldots, g_n \rangle$ for which $g_1 + \ldots + g_n = A$.

16. Interval case: definitions (cont-d)

- We say that the solution S is *fair* if the following two requirements are satisfied when $S(\langle[\underline{d}_1, \overline{d}_1], \ldots, [\underline{d}_n, \overline{d}_n]\rangle) = \langle g_1, \ldots, g_n\rangle$:
  - if $\underline{d}_i \le \underline{d}_j$ and $\overline{d}_i \le \overline{d}_j$, then $g_i \le g_j$;
  - $S(\langle[\underline{d}_1 + \underline{d}_2, \overline{d}_1 + \overline{d}_2], [\underline{d}_3, \overline{d}_3], \ldots, [\underline{d}_n, \overline{d}_n]\rangle) = \langle g_1 + g_2, g_3 \ldots, g_n\rangle$.
- We say that the solution S is *continuous*, if for every n:
  - if $\underline{d}_i^{(k)} \to d_i$ and $\overline{d}_i^{(k)} \to d_i$ for all i, $S(\langle[\underline{d}_1^{(k)}, \overline{d}_1^{(k)}], \ldots, [\underline{d}_n^{(k)}, \overline{d}_n^{(k)}]\rangle) = \langle g_1^{(k)}, \ldots, g_n^{(k)}\rangle$, and $g_i^{(k)} \to g_i$ for all i,
  - then $S(\langle[\underline{d}_1, \overline{d}_1], \ldots, [\underline{d}_n, \overline{d}_n]\rangle) = \langle g_1, \ldots, g_n\rangle$.

17. Main result

- **Proposition.** *For each solution S, the following two conditions are equivalent to each other:*
  - *the solution is fair and continuous,*
  - *for some $\alpha \in [0,1]$, the solution has the form $g_i = d_i \cdot (A/D)$, where*

$$d_i \stackrel{\text{def}}{=} \alpha \cdot \overline{d}_i + (1-\alpha) \cdot \underline{d}_i \text{ and } D \stackrel{\text{def}}{=} \alpha \cdot \overline{D} + (1-\alpha) \cdot \underline{D}.$$

- So, the solutions based on Hurwicz combinations $d_i = \alpha \cdot \overline{d}_i + (1-\alpha) \cdot \underline{d}_i$ are the only one which are fair (and continuous).
- **Proof.** It is east to check that the solution based on Hurwicz combination is fair and continuous.
- So, to complete the proof, it is sufficient to prove that every fair continuous solution S has this form.
- Indeed, let S be a fair and continuous solution.

18. Proof (cont-d)

- For every natural number N, we can consider the tuple

  $\langle [\underline{D}/N, \underline{D}/N], \ldots, [\underline{D}/N, \underline{D}/N], [0, (\overline{D} - \underline{D})/N], \ldots, [0, (\overline{D} - \underline{D})/N] \rangle.$

- This tuple consists of:
  - N degenerate debt intervals $[\underline{D}/N, \underline{D}/N]$ and
  - N intervals $[0, (\overline{D} - \underline{D})/N]$.
- By the first fairness requirement, since the debts d_i are the same for all first N creditors, the payments g_i should also be all equal:

$$g_1 = \ldots = g_N.$$

- Similarly, the payments to the last N creditors should be the same:

$$g_{N+1} = \ldots = g_{2N}.$$

19. Proof (cont-d)

- Let us take any two sequences of natural numbers $k_1, \ldots, k_n, \ell_1, \ldots, \ell_n$ for which $k_1 + \ldots + k_n = \ell_1 + \ldots + \ell_n = N$.
- Let us consider the tuple

$$\langle [k_1 \cdot (\underline{D}/N), k_1 \cdot (\underline{D}/N) + \ell_1 \cdot (\overline{D} - \underline{D})/N], \ldots,$$

$$[k_n \cdot (\underline{D}/N), k_n \cdot (\underline{D}/N) + \ell_n \cdot (\overline{D} - \underline{D})/N] \rangle.$$

- This tuple can be obtained from the above $2N$-tuple by adding up:
  - the first k_1 intervals from the first half and the first ℓ_1 intervals from the second half, then
  - the next k_2 intervals from the first half and the next ℓ_2 intervals from the second half, etc.

20. Proof (cont-d)

- So, due to the second fairness requirements, the resulting payment tuple $\langle g_1, \ldots, g_n \rangle$ can be obtained
  - from the tuple $\langle g_1, \ldots, g_1, g_{N+1}, \ldots, g_{N+1} \rangle$
  - by adding the corresponding payment terms.
- Thus, the payment tuple has the form

$$\langle k_1 \cdot g_1 + \ell_1 \cdot g_{N+1}, \ldots, k_n \cdot g_1 + \ell_n \cdot g_{N+1} \rangle.$$

- In other words:
  - for each debt interval

$$[\underline{d}_i, \overline{d}_i] = [k_i \cdot (\underline{D}/N), k_i \cdot (\underline{D}/N) + \ell_i \cdot (\overline{D} - \underline{D})/N],$$

  - the payment is equal to $g_i = k_i \cdot g_1 + \ell_i \cdot g_{N+1}$.
- Here, $\underline{d}_i = k_i \cdot (\underline{D}/N)$, so $k_i = \underline{d}_i \cdot (N/\underline{D})$.
- Similarly, $\overline{d}_i - \underline{d}_i = \ell_i \cdot ((\overline{D} - \underline{D})/N)$ so $\ell_i = (\overline{d}_i - \underline{d}_i) \cdot (N/(\overline{D} - \underline{D}))$.

21. Proof (cont-d)

- Substituting these expressions for k_i and ℓ_i into the above formula, we conclude that $g_i = a \cdot \underline{d}_i + b \cdot (\overline{d}_i - \underline{d}_i)$.
- Here we denoted $a \stackrel{\text{def}}{=} g_1 \cdot (N/\underline{D})$ and $b \stackrel{\text{def}}{=} g_{N+1} \cdot (N/(\overline{D} - \underline{D}))$.
- Thus, we have $g_i = b \cdot \overline{d}_i + (a - b) \cdot \underline{d}_i$.
- The first fairness requirement means that:
  - if $\overline{d}_i$ is larger then $\overline{d}_j$ while $\underline{d}_i = \underline{d}_j$,
  - then g_i should be larger (or the same) than g_j.
- This implies that $a \geq 0$.
- Similarly:
  - if $\underline{d}_i$ is larger then $\underline{d}_j$ while $\overline{d}_i = \overline{d}_j$,
  - then g_i should be larger (or the same) than g_j.
- This implies that $a - b \geq 0$.

22. Proof (cont-d)

- Let us denote the ratio b/a by α.
- Then, $b = a \cdot \alpha$ and $a - b = a \cdot (1 - \alpha)$.
- Thus, the above formula takes the form

$$g_i = a \cdot (\alpha \cdot \overline{d}_i + (1 - \alpha) \cdot \underline{d}_i).$$

- The sum of all the payments is equal to A, so

$$g_1 + \ldots + g_n = a \cdot (\alpha \cdot d_1 + (1 - \alpha) \cdot \underline{d}_1 + \ldots + \alpha \cdot d_n + (1 - \alpha) \cdot \underline{d}_n) =$$

$$a \cdot (\alpha \cdot (\underline{d}_1 + \ldots + \underline{d}_n) + (1 - \alpha) \cdot (\overline{d}_1 + \ldots + \overline{d}_n)) = a \cdot (\alpha \cdot \overline{D} + (1 - \alpha) \cdot \overline{D}) = a \cdot D.$$

- Hence $a = A/D$ and the above formula takes the desired form

$$g_i = (\alpha \cdot \overline{d}_i + (1 - \alpha) \cdot \underline{d}_i) \cdot (A/D).$$

23. Proof (cont-d)

- We have proved the desired equality for all the cases when for all the creditors i,
  - we have $\underline{d}_i = k_i \cdot (\underline{D}/N)$ for some integer k_i and
  - we have $\overline{d}_i - \underline{d}_i = \ell_i \cdot ((\overline{D} - \underline{D})/N)$ for some integer ℓ_i.
- Similarly to the proof of the previous Proposition:
  - any two real numbers can be thus approximated, and
  - the larger N, the more accurate the resulting approximation.
- Thus, due to continuity, in the limit $N \to \infty$, we have the desired equality for all possible values $\underline{d}_i$ and $\overline{d}_i$.
- The proposition is proven.

24. What if we have fuzzy uncertainty?

- For each creditor:
  - instead of a single interval,
  - we can have different intervals $[\underline{d}_i(\alpha), \overline{d}_i(\alpha)]$ containing d_i with different degrees of uncertainty $\alpha \in [0, 1]$.
- If we pick a narrower sub-interval, then:
  - we become less certain that d_i belongs to this sub-interval
  - than that it belongs to the original interval.
- Thus, the interval corresponding to a higher degree of uncertainty is a subset of the interval corresponding to a lower degree of uncertainty.
- Such a sequence of embedded intervals is, in effect, an equivalent representation of a so-called *fuzzy number*.
- The corresponding intervals are known as α-*cuts*.

25. What if we have fuzzy uncertainty (cont-d)

- In this case, to describe each creditor's debt:
  - instead of two values $\underline{d}_i$ and $\overline{d}_i$,
  - we need to describe infinitely many values $\underline{d}_i(\alpha)$ and $\overline{d}_i(\alpha)$ corresponding to different $\alpha \in [0, 1]$.
- The overall debt corresponding to different α can be obtained by adding all n debts:

$$\underline{D}(\alpha) = \underline{d}_1(\alpha) + \ldots + \underline{d}_n(\alpha) \text{ and } \overline{D}(\alpha) = \overline{d}_1(\alpha) + \ldots + \overline{d}_n(\alpha).$$

26. What if we have fuzzy uncertainty (cont-d)

- Arguments similar to the ones we used in the proof of Proposition 2 lead to a conclusion that:
  - a fair solution
  - is proportional to the linear combination d_i of these values:

$$g_i = d_i \cdot (A/D).$$

- Here, $d_i \stackrel{\text{def}}{=} \int (f_-(\alpha) \cdot \underline{d}_i(\alpha) + f_+(\alpha) \cdot \overline{d}_i(\alpha))\, d\alpha$ for some functions (maybe generalized functions) $f_\pm(\alpha)$, and

$$D \stackrel{\text{def}}{=} \int (f_-(\alpha) \cdot \underline{D}(\alpha) + f_+(\alpha) \cdot \overline{D}(\alpha))\, d\alpha.$$

27. What if we have probabilistic uncertainty?

- What if for each d_i, we only know the probability distribution?
- In this case, it makes sense to use the following additional requirement on the bankruptcy solutions:
  - if we repeat the same division situation several (N) times,
  - the payments in the resulting overall situation should be N times larger.
- In the overall situation, the debt amount D_i is equal to the sum of N independent equally distributed debt amounts: $D_i = d_i^{(1)} + \ldots + d_i^{(N)}$.
- According to the Large Numbers Theorem, for large N, the average $\dfrac{D_i}{N} = \dfrac{d_i^{(1)} + \ldots + d_i^{(N)}}{N}$ tends to the mean $E[d_i]$ as N increases.
- Thus, for large N, the sum is getting (relatively) closer and closer to a single value – N times the mean.

28. What if we have probabilistic uncertainty (cont-d)

- So, for large N, we have, in effect, the division problem in which:
  - instead of the original random variables,
  - we have N times their means.
- The payments in the original problem should be N times smaller.
- So, they should be simply equal to the division corresponding to the means.
- Thus, in the probabilistic case, we should simply:
  - compute the mean values $E[d_i]$ of the debt amount, and d
  - distribute the assets proportionally to these mean values:

$$g_i = \frac{E[d_i]}{E[d_1] + \ldots + E[d_n]} \cdot A.$$

29. Acknowledgments

- This work was supported in part by the National Science Foundation grants:
  - 1623190 (A Model of Change for Preparing a New Generation for Professional Practice in Computer Science), and
  - HRD-1834620 and HRD-2034030 (CAHSI Includes).
- It was also supported by the AT&T Fellowship in Information Technology.
- It was also supported by the program of the development of the Scientific-Educational Mathematical Center of Volga Federal District No. 075-02-2020-1478.