Why Quantiles Are a Good Description of Volatility in Economics: An Alternative Explanation

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1. Need to represent a random gain by a single number

- In economics, the outcomes of a decision are usually known with uncertainty.
- Based on the previous experience, for each possible decision, we can estimate the probability of different gains m.
- Thus, each possible decision can be characterised by a probability distribution on the set of possible gains m.
- This probability distribution can be described:
  - by a probability density function $\rho(m)$, or
  - by the cumulative distribution function $F(n) \stackrel{\text{def}}{=} \text{Prob}(m \le n)$.
- To select the best decision, we need to be able to compare every two possible decisions.
- For this purpose, we need to represent each possible decision by a single number.

2. Problem: what should this number be?

- How can we select this number?
- According to decision theory:
  - decisions of a rational person are equivalent to
  - selecting a decision that leads to the largest possible mean value of this person's utility.
- In the first approximation, utility is proportional to the gain.
- According to this logic, we should select a decision that leads to the largest possible value of the mean gain.

3. What we do in this talk

- In this paper, we show that a more appropriate decision is to select the decision with the largest possible value of the appropriate quantile.
- This provides an additional explanation for the fact that in econometrics:
  - an appropriate quantile – known as the Value at Risk (VaR),
  - is an accepted measure of the investment's volatility.

4. Analysis of the Problem and Its Resulting Formulation in Precise Terms

- Suppose that we represent a decision by a number n.
- Since the outcomes are random, the actual gain m will be, in general, different from n.
- How will this difference affect the decision maker?

5. Case when we gained more than expected

- Let us first consider the case when the actual gain m is larger than n.
- In this case, we can use the unexpected surplus $m - n$.
- For example, a person can take a trip, a company can buy some new equipment, etc.
- However, the value of this additional amount to the user is somewhat decreased by the fact that this amount was unexpected.
- For example, if a user plans a trip way beforehand, it is much cheaper than buying it in the last minute.
- If the company plans to buy an equipment some time ahead, it can negotiate a better price.

6. Case when we gained more than expected (cont-d)

- In all these cases:
  - in comparison to the user's value of each dollar of the expected amount n,
  - each dollar from the unexpected additional amount $m - n$ has a somewhat lower value, valued less by some coefficient $\alpha_+ > 0$.
- The loss of value for each dollar above the expected value is α_+.
- Thus, the overall loss corresponding to the whole unexpected amount $m - n$ is equal to $\alpha_+ \cdot (m - n)$.
- So, to get the overall user's value $v(m)$ of the gain m, we need to subtract this loss from m:

$$v(m) = m - \alpha_+ \cdot (m - n).$$

7. Case when we gained less than expected

- What if the actual gain m is smaller than n? In this case:
  - not only we lose the difference $n - m$ in comparison to what we expected, but
  - we lose some more.
- For example, since we expected the gain n, we may have already made some purchases for which we planned to pay from this amount.
- Since we did not get as much money as we expected, we need to borrow the missing amount of money.
- Since borrowing money comes with an interest, we thus lose – e.g., on this interest – some additional amount.
- In other words, for each dollar that we did not receive, we lose some additional amount.
- Let us denote this additional loss by $\alpha_- > 0$.

8. Case when we gained less than expected (cont-d)

- The overall additional loss to the user caused by the difference $n - m$ can be obtained by multiplying:
  - this difference by
  - the per-dollar loss α_-.
- So this loss is equal to $\alpha_- \cdot (n - m)$.
- The overall user's value $v(m)$ corresponding to the gain m can be thus obtained by subtracting this loss from the monetary amount m:

$$v(m) = m - \alpha_- \cdot (n - m).$$

9. Resulting optimization problem.

- As we have mentioned, a rational agent should maximize the expected value.
- So, a rational agent should select the value n that maximizes the expression

$$V \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \rho(m) \cdot v(m)\, dm =$$

$$\int_{-\infty}^{n} \rho(m) \cdot (m - \alpha_- \cdot (n - m))\, dm + \int_{n}^{\infty} \rho(m) \cdot (m - \alpha_+ \cdot (m - n))\, dm.$$

10. Solving the resulting optimization problem

- Differentiating the above expression with respect to the unknown value n and equating the resulting derivative to 0, we conclude that

$$-\alpha_- \cdot \int_{-\infty}^{n} \rho(m)\,dm + \alpha_+ \cdot \int_{n}^{\infty} \rho(m)\,dm = 0.$$

- Here, by definition of the probability density:

$$\int_{-\infty}^{n} \rho(m)\,dm = \text{Prob}(m \le n) = F(n) \text{ and}$$

$$\int_{n}^{\infty} \rho(m)\,dm = \text{Prob}(m \ge n) = 1 - F(n).$$

- Thus, the above equality takes the form

$$-\alpha_- \cdot F(n) + \alpha_+ \cdot (1 - F(n)) = 0, \text{ so}$$

$$(\alpha_- + \alpha_+) \cdot F(n) = \alpha_+ \text{ and hence } F(n) = \frac{\alpha_+}{\alpha_- + \alpha_+}.$$

11. Solving the resulting optimization problem (cont-d)

- This is exactly the quantile corresponding to
$$\tau = \frac{\alpha_+}{\alpha_- + \alpha_+}.$$
- Thus we arrive at the following conclusion.

12. Conclusion

- In the above natural optimization problem,
  - the optimal value n representing the random variable described by a cumulative distribution function $F(m)$ is
  - the quantile corresponding to the value

$$\tau = \frac{\alpha_+}{\alpha_- + \alpha_+}.$$

- This explains why quantiles (i.e., VaR) indeed work well in econometrics.

13. An interesting observation

- In econometrics, quantiles are not only used to describe the risk of different investments.
- They are also used to describe the dependence between different random variables – in the form of describing:
  - how the quantile of the dependent variable m
  - depends on the quantiles of the corresponding independent variables $x_1, \ldots, x_k$.
- In this technique – known as *quantile regression*:
  - for each value $\tau \in (0, 1)$
  - the τ-level quantile n of the random variable m is determined by minimizing the expression

$$I \stackrel{\text{def}}{=} (\tau - 1) \cdot \int_{-\infty}^{n} \rho(m) \cdot (m - n)\, dm + \tau \cdot \int_{n}^{\infty} \rho(m) \cdot (m - n)\, dm.$$

14. An interesting observation (cont-d)

- By comparing the formulas for V and I for the value τ determined by the above formula, we see that

$$V = \int_{-\infty}^{\infty} \rho(m) \cdot m\, dm - (\alpha_+ + \alpha_-) \cdot I.$$

- The first integral in this expression is just the expected value of the gain.
- This term does not depend on n at all.
- Thus, maximizing V is equivalent to minimizing the expression I.
- So:
  - the formal optimized expression used in quantile regression actually has a precise meaning:
  - it is linearly related to the expected utility of the user.

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