

# Why Decisions Based on the Results of Worst-Case, Most Realistic, and Best-Case Scenarios Work Well?

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## 1. Three scenarios: a brief reminder

- Often, we need to make a decision – about a plant, money investment, etc. infrastructure.
- To make a decision, it is important to predict the future state of the corresponding environment.
- The future state depends on many difficult-to-predict factors.
- E.g., the future of the investment is strongly affected by the central bank's interest rate.
- A common practice is:
  - to come up with three scenarios: the worst-case scenario, the most realistic scenario, and the best-case scenario, and
  - to make a decision by taking into account all three scenarios.

## 2. But why these three scenarios: formulation of the problem

- It make perfect sense to consider the most realistic scenario.
- It is probably, on average, the closest to what will happen in the future.
- But why it makes sense to consider the worst-case and the best-case scenarios is not clear.
- Both these scenarios represent idealized cases, cases that are practically impossible.
- The best-case scenario means that everything is perfectly beneficial to the company.
- It is possible that some favorable circumstances happen, but it is highly improbable that all circumstances are perfectly favorable.
- Similarly, the worst-case scenario means that everything is stacked against the company.
- It is possible that a company faces many challenges.

### 3. But why these three scenarios (cont-d)

- However, it is highly improbable that all the factors are against the company at the same time.
- From this viewpoint, it seems to be more appropriate:
  - instead of the unrealistic worst-case scenario,
  - to consider a more realistic scenario in between the worst case and the more realistic case.
- It also seems reasonable:
  - instead of the unrealistic best-case scenario,
  - to consider a more realistic scenario in between the best case and the more realistic case.
- However, in practice, people continue to use the worst-case and the best-case scenarios.

#### 4. But why these three scenarios (cont-d)

- It looks like:
  - the decisions based on these not very realistic scenarios,
  - in general, work better than decision made based on more realistic positive and negative scenarios.
- How can we explain this empirical fact?
- In this talk, we provide a possible explanation for the empirical success of using the worst-case and the best-case scenarios.

## 5. Numerical description

- To analyze the problem, let us describe it in precise mathematical terms.
- The effect of difficult-to-predict circumstances on the performance of our system can be usually described in numerical terms.
- For example, if we consider how resilient is a structure to possible earthquakes:
  - the best case is when the earthquake power stays at the same average level as now, and
  - the worst case when we have an earthquake of the power that is predicted to occur once every 100 years or so.
- We can analyze how resilient is the given company:
  - to the possible future shortage (and resulting high prices) of the materials it needs to function;
  - e.g., metal ore for metallurgical companies, oil for chemical plants, etc.

## 6. Numerical description (cont-d)

- In all these examples, the quality of the situation is described by some number  $x$ .
- Without losing generality, we can assume that smaller values of  $x$  correspond to more beneficial cases – as is the case of earthquakes and prices.
- Indeed, if larger values of  $x$  were better, we could simply consider  $-x$  instead of  $x$ .
- Under our assumption:
  - the worst-case scenario corresponds to the largest possible value of  $x$ , and
  - the best-case scenario corresponds to the smallest possible value of  $x$ .

## 7. Numerical description (cont-d)

- Let us denote:
  - the value corresponding to the most realistic scenario by  $\tilde{x}$ ,
  - the value corresponding to the worst-case scenario – i.e., to the largest possible value  $x$  – by  $\bar{x}$ , and
  - the value corresponding to the best-case scenario by  $\underline{x}$ .
- So, we have  $\underline{x} < \tilde{x} < \bar{x}$ .

## 8. What we want

- From the analysis of possible situations, we know the values  $\underline{x}$ ,  $\tilde{x}$ , and  $\bar{x}$ .
- Based on these three values, we need:
  - to come up with three values  $x_1 < x_2 < x_3$  for which we will analyze how the system reacts to the corresponding situation,
  - and then to make a decision based on the results of this analysis.
- In mathematical terms, we need to come up with a function that transforms a given triple  $(\underline{x}, \tilde{x}, \bar{x})$  into a new triple  $(x_1, x_2, x_3)$ .
- In the usual three-scenarios approach, we simply take  $x_1 = \underline{x}$ ,  $x_2 = \tilde{x}$ , and  $x_3 = \bar{x}$ .
- In the above-described seemingly more intuitive approach, we still take  $x_2 = \tilde{x}$ , but now:
  - instead of  $x_1 = \underline{x}$ , we select a value  $x_1$  for which  $\underline{x} < x_1 < \tilde{x}$ , and
  - instead of  $x_3 = \bar{x}$ , we select a value  $x_3$  for which  $\tilde{x} < x_3 < \bar{x}$ .

## 9. What we want (cont-d)

- Which of these two approaches is better?
- Which approach is optimal – i.e., better than all other approaches?
- To answer this question, we need to recall what we mean by optimal.

## 10. What do we mean by optimal: a seemingly natural approach and its limitations

- In many practical situations, we have an objective function  $f(z)$  that we are trying to maximize – such as profit for a company; in this case:
  - an alternative  $z$  is better than the alternative  $z'$  if  $f(z) > f(z')$ , and
  - an alternative  $z_{\text{opt}}$  is optimal if it has the largest possible value of this objective function, i.e., if  $f(z_{\text{opt}}) \geq f(z)$  for all alternatives  $z$ .
- In practice, however, this approach is unnecessarily limited.
- One of the problems is that often, there are several alternatives  $z$  that lead to the same value of the objective function.
- For example, we may have several alternatives that lead to the same value of the current year's profit.
- In this case, we need to come up with some other criterion  $g(z)$  that would enable us to choose one of these alternatives.

## 11. What do we mean by optimal (cont-d)

- So, in effect, the original optimality criterion is not final.
- In the profit example:
  - we can select an alternative that promises the largest profit in the following years,
  - or we can select an alternative that leads to the smallest impact on the environment.
- In such cases, we have a more complex selection criterion: namely,  $z$  is better than  $z'$  if:
  - either  $f(z) > f(z')$ ,
  - or  $f(z) = f(z')$  and  $g(z) > g(z')$ .
- If this still leaves us with several equally optimal alternatives, then we can use this non-uniqueness to optimize something else.
- We do it until we reach the situation when there is exactly one optimal alternative.

## 12. What do we mean by optimal: a general description

- How can we describe this general approach to describing optimality?
- In this general approach, all we have is the ability to compare two alternatives.
- In other words, for some (or even all) pairs  $(z, z')$  we can decide:
  - whether  $z$  is better than  $z'$ ; we will denote this by  $z \succ z'$ ,
  - or  $z'$  is better than  $z$ , i.e.,  $z' \succ z$ ,
  - or  $z$  and  $z'$  have the same value to us; we will denote this by  $z \sim z'$ .
- Of course, the results of these comparisons should be consistent.
- E.g., if  $z$  is better than  $z'$ , and  $z'$  is better than  $z''$ , then  $z$  should be better than  $z''$ .
- So, we arrive at the following definitions.

### 13. Definition

- Let  $A$  be a set; its elements will be called *alternatives*.
- By an *optimality criterion*, we mean a pair of binary relations  $(\succ, \sim)$  that satisfies the following properties for all possible  $z, z'$  and  $z''$ :
  - if  $z \succ z'$  and  $z' \succ z''$ , then  $z \succ z''$ ;
  - if  $z \succ z'$  and  $z' \sim z''$ , then  $z \succ z''$ ;
  - if  $z \sim z'$  and  $z' \succ z''$ , then  $z \succ z''$ ;
  - if  $z \sim z'$  and  $z' \sim z''$ , then  $z \sim z''$ ;
  - if  $z \sim z'$ , then  $z' \sim z$ ;
  - if  $z \succ z'$ , then we cannot have  $z \sim z'$ ; and
  - for all  $z$ , we have  $z \sim z$ .

## 14. Definition

- We say that an alternative  $z_{\text{opt}}$  is *optimal* with respect to the optimality criterion  $(\succ, \sim)$  if for every alternative  $z$ , we have:

either  $z_{\text{opt}} \succ z$  or  $z_{\text{opt}} \sim z$ .

- We say that the optimality criterion  $(\succ, \sim)$  is *final* if there exists exactly one alternative which is optimal with respect to this criterion.

## 15. We can have different scales for describing the situation

- To understand this, let us go back to our two examples: earthquakes and shortage of materials.
- For earthquakes, there are different scales for describing this power.
- We can describe it in Watts, or in Richer scale, which is proportional to the logarithm of the earthquake's power in Watts.
- For possible shortages:
  - we can measure the shortage by an increase in price,
  - or we can measure it by a decrease in the overall production of this material.

These two scales are related, but also not linearly.

- In many situations, we have different scales for measuring the same situation, scales that non-linearly depend on each other.

## 16. We can have different scales (cont-d)

- The only thing these scales have in common is that they are monotonic, i.e., either increasing or decreasing with respect to each other:
  - increasing means that an increase in one scale always means increase in another one, and
  - decreasing means that an increase in one scale always means decrease in another one.
- Since we agreed to limit ourselves to scales in which smaller values are better, we only need to consider increasing scales.

## 17. It is reasonable to require scale-invariance

- Our goal is to find an optimal mapping from triples to triples.
- For this purpose, we need to be able to compare different mappings of this type.
- So, we must have a final optimality criterion on the set of all such mappings.
- Since we can use different scales, it makes sense to require that the relative qualities of a mapping not depend on the scale:
  - If one mapping is better than the other if we use the original scale,
  - then it should be still better if we change the scale – since both scales describe the exact same practical situation.
- Let us describe this reasonable requirement in precise terms.

## 18. Definition

- By a *mapping*, we mean a function  $F$  that maps ordered triples  $\underline{x} < \tilde{x} < \bar{x}$  into ordered triples  $x_1 < x_2 < x_3$ .
- By a *re-scaling*, we means a strictly increasing 1-1 function  $r(x)$ .
- By the *result*  $r(F)$  of applying a re-scaling  $r(x)$  to the mapping  $F$ , we mean the following procedure:
  - first, we re-scale all the values from the original triple  $z = (\underline{x}, \tilde{x}, \bar{x})$ , i.e., come up with re-scaled triple  $r(z) \stackrel{\text{def}}{=} (r(\underline{x}), r(\tilde{x}), r(\bar{x}))$ ;
  - then, we apply  $F$  to the re-scaled input, resulting in the triple  $F(r(z))$ ;
  - and finally, we re-scale all the values from the new triple  $F(r(z))$  back into the original scale – by using the inverse function  $r^{-1}$ :

$$(r(F))(x) \stackrel{\text{def}}{=} r^{-1}(F(r(z))).$$

## 19. Example

- Suppose that we selected a function that maps the original triple  $z = (\underline{x}, \tilde{x}, \bar{x})$  into a new triple

$$F(z) = \left( x_1 = \frac{\underline{x} + \tilde{x}}{2}, x_2 = \tilde{x}, x_3 = \frac{\tilde{x} + \bar{x}}{2} \right).$$

- Let us take, as the re-scaling function, the function  $r(x) = \ln(x)$  from our earthquake example.

## 20. Example (cont-d)

- Then, if we apply the same mapping in the new logarithmic scale, this is equivalent to the following mapping in the original scale:
  - first, we transform the original values  $\underline{x} < \tilde{x} < \bar{x}$  into the new logarithmic scale; this way, we get new values

$$\underline{X} = \ln(\underline{x}), \quad \tilde{X} = \ln(\tilde{x}), \quad \bar{X} = \ln(\bar{x});$$

- then, by applying the same mapping to the new scale, resulting in the new tuple:

$$\left( X_1 = \frac{\underline{X} + \tilde{X}}{2}, X_2 = \tilde{X}, X_3 = \frac{\tilde{X} + \bar{X}}{2} \right);$$

- finally, we transform the values  $X_1$ ,  $X_2$ , and  $X_3$  into the original scale, by using the inverse transformation  $r^{-1}(x) = \exp(x)$ .

## 21. Example (cont-d)

- As a result, in the original scale, we get

$$x_1 = \exp\left(\frac{\underline{X} + \tilde{X}}{2}\right) = \exp\left(\frac{\ln(\underline{x}) + \ln(\tilde{x})}{2}\right) = \sqrt{\underline{x} \cdot \tilde{x}};$$

$$x_2 = \exp(\tilde{X}) = \exp(\ln(\tilde{x})) = \tilde{x};$$

$$x_3 = \exp\left(\frac{\tilde{X} + \bar{X}}{2}\right) = \exp\left(\frac{\ln(\tilde{x}) + \ln(\bar{x})}{2}\right) = \sqrt{\tilde{x} \cdot \bar{x}}.$$

- These three formulas describe, in this case, the result  $r(F)$  is applying the logarithmic re-scaling  $r(x) = \ln(x)$  to the above mapping  $F(z)$ .

## 22. Definition

- We say that the optimality criterion  $(\succ, \sim)$  on the set of all mappings is *scale-invariant* if
  - for every two mappings  $F$  and  $F'$  and for each re-scaling  $r$ ,
  - the following two conditions are satisfied.
    1. The mapping  $F$  is better than  $F'$  if and only if  $r(F)$  is better than  $r(F')$ :  $F \succ F' \Leftrightarrow r(F) \succ r(F')$ ;
    2. The mapping  $F$  is of the same quality as  $F'$  if and only if  $r(F)$  is of the same quality as  $r(F')$ :  $F \sim F' \Leftrightarrow r(F) \sim r(F')$ .

## 23. Main Result

- Now, we can formulate our main result.
- This result explains the empirical effectiveness of using the best-case and the worst-case scenarios in decision making.
- *Let  $(\succ, \sim)$  be a final scale-invariant optimality criterion on the set of all possible mapping.*
- *Then, the mapping that is optimal with respect to this criterion maps the original triple  $(\underline{x}, \tilde{x}, \bar{x})$  into itself, i.e., generates the same triple*

$$(x_1 = \underline{x}, x_2 = \tilde{x}, x_3 = \bar{x}).$$

## 24. Proof of the Main Result

- Let us first prove that the optimal mapping  $F_{\text{opt}}$  is itself scale-invariant, i.e., that  $r(F_{\text{opt}}) = F_{\text{opt}}$  for all re-scalings  $r(x)$ .
- Indeed, let us fix a re-scaling  $r(x)$  and let us prove that for this re-scaling, we indeed have  $r(F_{\text{opt}}) = F_{\text{opt}}$ .
- Since the mapping  $F_{\text{opt}}$  is optimal, it means that it is better than or of the same quality than any other mapping  $F$ .
- In particular, for every mapping  $F$ , we have either  $F_{\text{opt}} \succ r^{-1}(F)$  or  $F_{\text{opt}} \sim r^{-1}(F)$ .
- Due to scale-invariance:

– the condition  $F_{\text{opt}} \succ r^{-1}(F)$  implies that

$$r(F_{\text{opt}}) \succ r(r^{-1}(F)) = F, \text{ and}$$

– the condition  $F_{\text{opt}} \sim r^{-1}(F)$  implies that

$$r(F_{\text{opt}}) \sim r(r^{-1}(F)) = F.$$

## 25. Proof (cont-d)

- Thus, for every mapping  $F$ , we have:

either  $r(F_{\text{opt}}) \succ F$  or  $r(F_{\text{opt}}) \sim F$ .

- By definition of an optimal alternative, this means that the mapping  $r(F_{\text{opt}})$  is optimal.
- However, since the optimality criterion is final, this means that there is only one optimal mapping.
- Thus indeed  $r(F_{\text{opt}}) = F_{\text{opt}}$ .

## 26. Proof (cont-d)

- Let  $z = (\underline{x}, \tilde{x}, \bar{x})$  be any triple.
- Let us denote the components of the triple  $F(z)$  by  $(x_1, x_2, x_3)$ .
- Let us prove, by contradiction, that each of the three components  $x_i$  coincides with one of the elements of the original tuple.
- Indeed, suppose that one of these components  $x_{i_0}$  is different from all three values from the original tuple.
- Then, if we change  $x_{i_0}$  to a sufficiently close number  $x_{i_0} + \varepsilon$  with small  $\varepsilon > 0$ , we can still preserve the order of the values  $\underline{x}$ ,  $\tilde{x}$ ,  $\bar{x}$ , and  $x_i$ .
- We can then form a piece-wise linear function  $r(x)$  for which:

$$r(\underline{x}) = \underline{x}, r(\tilde{x}) = \tilde{x}, r(\bar{x}) = \bar{x}, r(x_{i_0}) = x_{i_0} + \varepsilon, \text{ and} \\ r(x_i) = x_i \text{ for all } i \neq i_0.$$

- Since the optimal mapping is scale-invariant, from  $F(\underline{x}, \tilde{x}, \bar{x}) = (x_1, x_2, x_3)$ , we can conclude that

$$F(r(\underline{x}), r(\tilde{x}), r(\bar{x})) = (r(x_1), r(x_2), r(x_3)).$$

## 27. Proof (cont-d)

- Since for all three components of the input triple, the mapping  $r(x)$  does not change anything, we conclude that

$$F(\underline{x}, \tilde{x}, \bar{x}) = (r(x_1), r(x_2), r(x_3)).$$

- Thus,  $(x_1, x_2, x_3) = (r(x_1), r(x_2), r(x_3))$ , but this is not true since  $r(x_{i_0}) = x_{i_0} + \varepsilon \neq x_{i_0}$ .
- The contradiction proves that our assumption is wrong.
- So, each component  $x_i$  of the output  $(x_1, x_2, x_3)$  with  $x_1 < x_2 < x_3$  is indeed equal to one of the components of the input  $(\underline{x}, \tilde{x}, \bar{x})$  for which

$$\underline{x} < \tilde{x} < \bar{x}.$$

## 28. Proof (cont-d)

- Three different components  $x_i$  are different, and each of them is equal to one of the three components of the input.
- Thus, every input component is equal to one of the output components.
- Otherwise, if one of the input components was not equal to one of  $x_i$ 's:
  - then two of the output components should be equal to the same input components and thus, equal to each other,
  - but all  $x_i$ 's are different.

## 29. Proof (cont-d)

- Thus:
  - the smallest  $x_1$  of the output components is equal to the smallest input component  $\underline{x}$ ,
  - the in-between output component  $x_2$  is equal to the in-between input component  $\tilde{x}$ , and
  - the largest  $x_3$  of the output components is equal to the largest input component  $\bar{x}$ .
- The Proposition is proven.

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