On Chaos-Related Approach to Dynamics of Exchange Rates: From-First-Principles Derivation of Embrechts et al. Empirical Model

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1. Chaotic systems are ubiquitous

- Many real-life phenomena show chaotic behavior – when:
  - the corresponding dynamical systems are largely deterministic, but
  - due to the systems' complexity, we can only make probabilistic long-term predictions.

2. Chaotic behavior can also be observed in finance

- Finance is one of the areas where chaotic behavior has been observed.
- Applications of chaos theory to finance started with a pioneering 1993 book.
- This book showed, in particular, that chaotic behavior can be observed in the dynamics of exchange rates S_t.
- This dynamics is described by an empirical formula

$$S_{t+1} = c_t \cdot S_t^{\alpha_0} \cdot S_{t-1}^{\alpha_1}.$$

- Here c_t are behavioral variables that affect the exchange rates.
- If we take logarithms of both sides of the above equality, we get linear regression:

$$\ln(S_{t+1}) = \ln(c_t) + \alpha_0 \cdot \ln(S_t) + \alpha_1 \cdot \ln(S_{t-1}).$$

3. A natural question

- How can we explain the chaos-related empirical model for the dynamics of exchange rates?
- In this paper, we provide the from-first-principles derivation of this model.

4. Analysis of the problem

- We know the current exchange rate S_k and the past exchange rates $S_{t-1}, \ldots, S_{t-k}$.
- Based on this information, we want to predict the exchange rate S_{t+1} at the next moment of time.
- In the ideal deterministic case, the future value x_{t+1} of a quantity x is uniquely determined by:
  - its current value x_t, and
  - its past values $x_{t-1} \ldots, x_{t-k}$.
- In this case:
  - the only thing we need to do to predict the future values is
  - to find a function $x_{t+1} = f(x_t, x_{t-1}, \ldots, x_{t-k})$ that best describes the corresponding dynamics.

5. Specifics of predicting exchange rate

- Of course, for exchange rates:
  - even when we have all the previous values of the exchange rate,
  - and even all possible information about the given country's economy,
  - we cannot uniquely predict the future exchange rate.
- Indeed, the future exchange rate also depends on the fluctuations of the currency to which we are comparing the analyzed currency.
- It may be US Dollar or European Union's Euro.
- These fluctuations cannot be predicted based only on the given country's economy.

6. Specifics of predicting exchange rate (cont-d)

- Thus:
  - when the value of the compared-to currency decreases by a factor of c,
  - then the exchange rate between the analyzed country's currency and the compared-to currency is multiplied by c.
- Thus, in this case, we cannot have a *single* prediction function $f(x_t, x_{t-1}, \ldots, x_{t-k})$.
- We can only have a *family* of prediction functions

$$\{c \cdot f(x_t, x_{t-1}, \ldots, x_{t-k})\}_c.$$

- So, the question is: which of such families should we select?

7. Comment

- As we have mentioned, we cannot predict the future exchange rate.
- This does not mean that the above formula is useless.
- Indeed, the coefficient c is the same for all the countries with similar – and related – economies.
- Thus:
  - while we cannot predict the exchange rate of a *single* country,
  - we can predict the *ratio* of the exchange rates of two similar countries.

8. Let us take into account that we can use different compared-to currencies

- We have exchange rates with respect to US dollar, with respect to Euro, with respect to Japanese Yen, etc.
- It is therefore desirable to have a family of functions that should be useful for predicting *all* these different exchange rates.
- Let us describe this requirement in precise terms.
- Suppose that we have a family of functions that describes the dynamics of the exchange rate with respect to US Dollar.
- For this family, for each moment of time t, we should have

$$S_{t+1} = c_t \cdot f(S_t, S_{t-1}, \ldots, S_{t-k}).$$

- Suppose now that:
  - instead of comparing the analyzed currency with US Dollars,
  - we compare it with Euros (or with Japanese Yen).

9. Let us take into account that we can use different compared-to currencies (cont-d)

- To get an exchange rate between the analyzed currency and Euros, we need to multiply two values:
  - the exchange rate between the analyzed currency and US Dollar, and
  - the exchange C_t rate between US Dollar and the new compared-to-currency:

$$S'_t \stackrel{\text{def}}{=} \frac{\text{analyzed currency}}{\text{Euro}} = \frac{\text{analyzed currency}}{\text{US Dollar}} \cdot \frac{\text{US Dollar}}{\text{Euro}} = S_t \cdot C_t.$$

- We consider the case when the relation between the two compared-to currencies is reasonably stable.

10. Let us take into account that we can use different compared-to currencies (cont-d)

- In this case, all the values C_i are around some constant C, with relative difference from C not exceeding some small number $\varepsilon > 0$:

$$1 - \varepsilon \le \frac{C_i}{C} \le 1 + \varepsilon.$$

- In this case, for the new exchange rates S'_t, we should have a similar dependence $S'_{t+1} = c'_t \cdot f(S'_t, S'_{t-1}, \ldots, S'_{t-k})$ for some c'_t.
- As we have mentioned, the above formula holds not only for the given country.
- It holds for all the countries with a similar economy.
- Similarly, the formula for S'_{t+1} should be valid for all these countries.

11. Let us take into account that we can use different compared-to currencies (cont-d)

- This means that the value c'_t should depend only on the two compared-to currencies.
- The values c'_t should not depend on the present and past exchange rates specific for each of the analyzed countries.
- Now, we are ready to describe this requirement in precise terms.

12. Definitions

- Let $k \geq 0$ be a natural number.
- By a *k-prediction function*, we mean a measurable function $f(S_t, S_{t-1}, \ldots S_{t-k})$.
- We say that a prediction function $f(S_t, S_{t-1}, \ldots S_{t-k})$ is *consistent* if there exists a positive number $\varepsilon > 0$ such that:
  - for every tuple $(c_t, C, C_{t+1}, C_t, C_{t-1}, \ldots, C_{t-k})$ that satisfies the ε-condition,
  - there exists a value c'_t for which,
  - once we have values S_i that satisfy the equality

$$S_{t+1} = c_t \cdot f(S_t, S_{t-1}, \ldots, S_{t-k}),$$

  - then the values $S'_i = C_i \cdot S_i$ satisfy the equality

$$S'_{t+1} = c'_t \cdot f(S'_t, S'_{t-1}, \ldots, S'_{t-k}).$$

13. Main result and its consequences

- *A prediction function is consistent if and only if it has the form*

$$f(S_t, S_{t-1}, \ldots, S_{t-k}) = \alpha \cdot S_t^{\alpha_0} \cdot S_{t-1}^{\alpha_1} \cdot \ldots S_{t-k}^{\alpha_k}.$$

- In particular, for $k = 1$, we get the desired theoretical explanation for the above-mentioned empirical model.
- For $k > 1$, we get:
  - a natural extension of this model
  - to the case when we take into account that the exchange rate may also dependent on the exchange rate several years ago.
- Similarly to the case $k = 1$:
  - if we take logarithms of both sides of the main equality,
  - then we get a linear regression:

$$\ln(S_{t+1}) = \ln(\alpha) + \alpha_0 \cdot \ln(S_t) + \alpha_1 \cdot \ln(S_{t-1}) + \ldots + \alpha_k \cdot \ln(S_{t-k}).$$

14. Proof, Part 1

- It is easy to prove that every prediction function of the given type is consistent.
- Indeed, in this case, the original equality takes the form

$$S_{t+1} = c_t \cdot \alpha \cdot S_t^{\alpha_0} \cdot S_{t-1}^{\alpha_1} \cdot \ldots S_{t-k}^{\alpha_k}.$$

- The desired equality takes the following form:

$$S'_{t+1} = c'_t \cdot \alpha \cdot (S'_t)^{\alpha_0} \cdot (S'_{t-1})^{\alpha_1} \cdot \ldots (S'_{t-k})^{\alpha_k}.$$

- Substituting the expression $S'_i = C_i \cdot S_i$ into the last formula, we get

$$C_{t+1} \cdot S_{t+1} = c'_t \cdot \alpha \cdot (C_t \cdot S_t)^{\alpha_0} \cdot (C_{t-1} \cdot S_{t-1})^{\alpha_1} \cdot \ldots (C_{t-k} \cdot S_{t-k})^{\alpha_k}.$$

- Here, $(C_i \cdot S_i)^{\alpha_i} = C_i^{\alpha_i} \cdot S_i^{\alpha_i}$.
- Let us substitute these expressions into the formula for s'_{t+1}.
- If we place together all the terms proportional to the powers of S_i, we conclude that $C_{t+1} \cdot S_{t+1} = c'_t \cdot \alpha \cdot C' \cdot S_t^{\alpha_0} \cdot S_{t-1}^{\alpha_1} \cdot \ldots S_{t-k}^{\alpha_k}$.

15. Proof, Part 1 (cont-d)

- Here we denoted $C' \stackrel{\text{def}}{=} C_t^{\alpha_0} \cdot C_{t-1}^{\alpha_1} \cdot \ldots \cdot C_{t-k}^{\alpha_k}$.
- Thus, the desired formula is equivalent to

$$S_{t+1} = \frac{c_t'}{C_{t+1}} \cdot \alpha \cdot C' \cdot S_t^{\alpha_0} \cdot S_{t-1}^{\alpha_1} \cdot \ldots S_{t-k}^{\alpha_k}.$$

- The desired equality turns into the known equality when

$$\frac{c_t'}{C_{t+1}} \cdot C' = c_t.$$

- To attain this quality, we need to take $c_t' = \dfrac{C_{t+1} \cdot c_t}{C'}$.
- So, for this c_t', the original equality indeed implies the new one.

16. Proof, Part 2

- Let us prove that, vice versa, every consistent prediction function has the desired form.
- Indeed, for $C = 1$, substituting $S'_i = C_i \cdot S_i$ into the new equality, we conclude that

$$C_{t+1} \cdot S_{t+1} = c'_t \cdot f(C_t \cdot S_t, C_{t-1} \cdot S_{t-1}, \ldots, C_{t-k} \cdot S_{t-k}).$$

- Substituting the expression for S_{t+1} from the original formula, we conclude that

$$C_{t+1} \cdot c_t \cdot f(S_t, S_{t-1}, \ldots, S_{t-k}) = c'_t \cdot f(C_t \cdot S_t, C_{t-1} \cdot S_{t-1}, \ldots, C_{t-k} \cdot S_{t-k}).$$

- This implies that:

$$f(C_t \cdot S_t, C_{t-1} \cdot S_{t-1}, \ldots, C_{t-k} \cdot S_{t-k}) =$$

$$g(c_t, C_{t+1}, C_t, \ldots, C_{t-k}) \cdot f(S_t, S_{t-1}, \ldots, S_{t-k}).$$

- Here we denoted $g(c_t, C_{t+1}, C_t, \ldots, C_{t-k}) \stackrel{\text{def}}{=} \dfrac{C_{t+1}}{c'_t(c_t, C_t, \ldots, C_{t-k})}$.

17. Proof, Part 2 (cont-d)

- The above formula only holds for the values C_i for which

$$1 - \varepsilon \le C_i \le 1 + \varepsilon.$$

- However, if we have $C_i'' = C_i' \cdot C_i$, where $1 - \varepsilon \le C_i, C_i' \le 1 + \varepsilon$, then we get a similar equality:

$$f(C_t' \cdot (C_t \cdot S_t), C_{t-1}' \cdot (C_{t-1} \cdot S_{t-1}), \ldots, C_{t-k}' \cdot (C_{t-k} \cdot S_{t-k})) =$$

$$g(c_t, C_{t+1}', C_t', \ldots, C_{t-k}') \cdot f(C_t \cdot S_t, C_{t-1} \cdot S_{t-1}, \ldots, C_{t-k} \cdot S_{t-k}).$$

- Substituting the expression for $f(C_t \cdot S_t, C_{t-1} \cdot S_{t-1}, \ldots, C_{t-k} \cdot S_{t-k})$ into the right-hand side of this formula, we get

$$f(C_t'' \cdot S_t, C_{t-1}'' \cdot S_{t-1}, \ldots, C_{t-k}'' \cdot S_{t-k}) =$$

$$g(c_t, C_{t+1}'', C_t'', \ldots, C_{t-k}'') \cdot f(S_t, S_{t-1}, \ldots, S_{t-k}), \text{ where}$$

$$g(c_t, C_{t+1}'', C_t'', \ldots, C_{t-k}'') \stackrel{\text{def}}{=}$$

$$g(c_t, C_{t+1}', C_t', \ldots, C_{t-k}') \cdot g(c_t, C_{t+1}, C_t, \ldots, C_{t-k}).$$

18. Proof, Part 2 (cont-d)

- In other words, we can have the above formula:
  - not only for the tuple of values C_i between $1-\varepsilon$ and $1+\varepsilon$,
  - but also for componentwise products of such tuples.
- Similarly, we can the same result for componentwise product of any finite number of such tuples.
- For each positive number C_i, the value $C_i^{1/n}$ tends to $C_i^0 = 1$ when n tends to infinity.
- Thus, for any tuple of positive numbers, and for sufficiently large n, we have $1-\varepsilon \le C_i^{1/n} \le 1+\varepsilon$ for all i.
- For the resulting tuple of n-th order roots, we can get the above formula.
- Thus, we can get it for the componentwise product of n such tuples – i.e., to the original tuple of the values C_i.

19. Proof, Part 2 (cont-d)

- So, the below equality holds for all possible positive tuples S_i and C_i:

$$f(C_t \cdot S_t, C_{t-1} \cdot S_{t-1}, \ldots, C_{t-k} \cdot S_{t-k}) =$$

$$g(c_t, C_{t+1}, C_t, \ldots, C_{t-k}) \cdot f(S_t, S_{t-1}, \ldots, S_{t-k}).$$

- It is known that all measurable functions that satisfy this equality for all positive tuples have the desired form.
- The proposition is proven.

20. Acknowledgments

This work was supported in part:

- by the US National Science Foundation grants:
  - 1623190 (A Model of Change for Preparing a New Generation for Professional Practice in Computer Science),
  - HRD-1834620 and HRD-2034030 (CAHSI Includes),
  - EAR-2225395 (Center for Collective Impact in Earthquake Science C-CIES),
- by the AT&T Fellowship in Information Technology, and
- by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) Focus Program SPP 100+ 2388, Grant Nr. 501624329,