Towards Fast Algorithms for Fuzzy Data Processing: Type 1, Type 2, and Beyond or How Interval Ideas Travelled from Warszawa, Tokyo, and California Back to Warszawa

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Many of these results come from joint papers
with Andrzej Pownuk

1. Outline

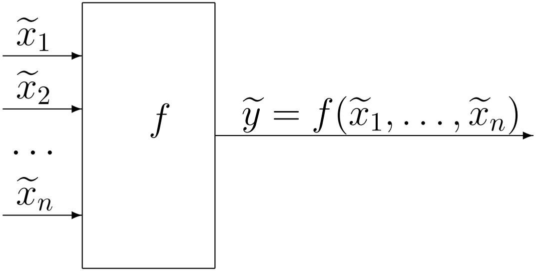
- From the mathematical viewpoint, we can use Zadeh's extension principle to process fuzzy data.
- However, a direct implementation of Zadeh's extension principle often requires too many computational steps.
- A known way to speed up computations is to use interval computations on α-cuts.
- We show that in many cases, we can further reduce computation time.
- The need to decrease computation time is even more important for type-2 fuzzy sets.
- They require even more computations; we show that for type-2, a significant speed up is also possible.
- We also extend the speed-up beyond the min t-norm.

Part I

Fuzzy Data Processing: What Is the Problem, and Which Algorithms Are Available for Solving This Problem

2. Why Data Processing and Knowledge Processing Are Needed in the First Place

- *Problem:* some quantities y are difficult (or impossible) to measure or estimate directly.
- *Solution:* indirect measurements or estimates



- *Fact:* estimates $\widetilde{x}_i$ are approximate.
- *Question:* how approximation errors $\Delta x_i \stackrel{\text{def}}{=} \widetilde{x}_i - x_i$ affect the resulting error $\Delta y = \widetilde{y} - y$?

3. From Probabilistic to Interval Uncertainty

- Manufacturers of MI provide us with bounds Δ_i on measurement errors: $|\Delta x_i| \le \Delta_i$.
- Thus, we know that $x_i \in [\widetilde{x}_i - \Delta_i, \widetilde{x}_i + \Delta_i]$.
- Often, we also know probabilities, but in 2 cases, we don't:
  - cutting-edge measurements;
  - cutting-cost manufacturing.
- In such situations:
  - we know the intervals $[\underline{x}_i, \overline{x}_i] = [\widetilde{x}_i - \Delta_i, \widetilde{x}_i + \Delta_i]$ of possible values of x_i, and
  - we want to find the range of possible values of y:

$$\mathbf{y} = [\underline{y}, \overline{y}] = \{f(x_1, \ldots, x_n) : x_1 \in [\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_n, \overline{x}_n]\}.$$

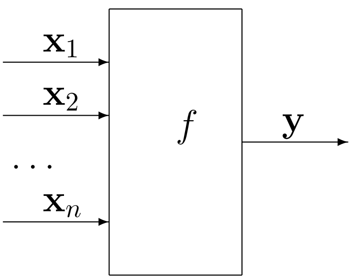
4. Main Problem of Interval Computations

We are given:

- an integer n;
- n intervals $\mathbf{x}_1 = [\underline{x}_1, \overline{x}_1]$, …, $\mathbf{x}_n = [\underline{x}_n, \overline{x}_n]$, and
- an algorithm $f(x_1, \ldots, x_n)$ which transforms n real numbers into a real number $y = f(x_1, \ldots, x_n)$.

We need to compute the endpoints $\underline{y}$ and $\overline{y}$ of the interval

$$\mathbf{y} = [\underline{y}, \overline{y}] = \{f(x_1, \ldots, x_n) : x_1 \in [\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_n, \overline{x}_n]\}.$$



Outline
Fuzzy Data Processing
Interval Computations
Reduction to Interval …
Measurement and …
Further Speed-Up
Type-2 Fuzzy Case
Beyond min t-Norm
Reduction to Informal …
Home Page
Title Page
◀◀ ▶▶
◀ ▶
Page 6 of 45
Go Back
Full Screen
Close
Quit

5. Interval Computations: A Brief History

- *Origins:* Archimedes (Ancient Greece)
- *Pioneers:* Mieczyslaw Warmus (Poland), Teruo Sunaga (Japan), Raymond Moore (USA), 1956–59
- *First boom:* early 1960s.
- *First challenge:* taking interval uncertainty into account when planning spaceflights to the Moon.
- *Current applications* (sample):
  - design of elementary particle colliders: Martin Berz, Kyoko Makino (USA)
  - will a comet hit the Earth: Berz, Moore (USA)
  - robotics: Jaulin (France), Neumaier (Austria)
  - chemical engineering: Mark Stadtherr (USA)

6. Need to Process Fuzzy Uncertainty

- In many practical situations, we only have expert estimates for the inputs x_i.
- Sometimes, experts provide guaranteed bounds on x_i, and even the probabilities of different values.
- However, such cases are rare.
- Usually, the experts' opinion is described by (imprecise, "fuzzy") words from natural language.
- Example: the value x_i of the i-th quantity is approximately 1.0, with an accuracy most probably about 0.1.
- Based on such "fuzzy" information, what can we say about $y = f(x_1, \ldots, x_n)$?
- The need to process such "fuzzy" information was first emphasized in the early 1960s by L. Zadeh.

7. How to Describe Fuzzy Uncertainty: Reminder

- In Zadeh's approach, we assign:
  - to each number x_i,
  - a degree $m_i(x_i) \in [0,1]$ with which x_i is a possible value of the i-th input.
- In most practical situations, the membership function:
  - starts with 0,
  - continuously $\uparrow$ until a certain value,
  - and then continuously $\downarrow$ to 0.
- Such membership function describe usual expert's expressions such as "small", "$\approx a$ with an error $\approx \sigma$".
- Membership functions of this type are actively used in expert estimates of number-valued quantities.
- They are thus called *fuzzy numbers*.

8. Processing Fuzzy Data: Formulation of the Problem

- We know an algorithm $y = f(x_1, \ldots, x_n)$ that relates:
  - the value of the desired difficult-to-estimate quantity y with
  - the values of easier-to-estimate auxiliary quantities $x_1, \ldots, x_n$.
- We also have expert knowledge about each of the quantities x_i.
- For each i, this knowledge is described in terms of the corresponding membership function $m_i(x_i)$.
- Based on this information, we want to find the membership function $m(y)$ which describes:
  - for each real number y,
  - the degree of confidence that this number is a possible value of the desired quantity.

9. Towards Solving the Problem

- Intuitively, y is a possible value of the desired quantity if for some values $x_1, \ldots, x_n$:
  - x_1 is a possible value of the 1st input quantity,
  - and x_2 is a possible value of the 2nd input quantity,
  - $\ldots$,
  - and $y = f(x_1 \ldots, x_n)$.
- We know:
  - that the degree of confidence that x_1 is a possible value of the 1st input quantity is equal to $m_1(x_1)$,
  - that the degree of confidence that x_2 is a possible value of the 2nd input quantity is equal to $m_2(x_2)$,
  - etc.
- The degree of confidence $d(y, x_1, \ldots, x_n)$ in an equality $y = f(x_1 \ldots, x_n)$ is, of course, 1 or 0.

10. Towards Solving the Problem (cont-d)

- The simplest way to represent "and" is to use min.
- Thus, for each combination of values $x_1, \ldots, x_n$, the degree of confidence d in a composite statement

  "x_1 is a possible value of the 1st input quantity, and x_2 is a possible value of the 2nd input quantity, ..., and $y = f(x_1 \ldots, x_n)$"

  is equal to

  $$d = \min(m_1(x_1), m_2(x_2), \ldots, d(y, x_1, \ldots, x_n)).$$

- We can simplify this expression if we consider two possible cases:
  - when $y = f(x_1 \ldots, x_n)$, we get

    $$d = \min(m_1(x_1), m_2(x_2), \ldots, d(y, x_1, \ldots, x_n));$$

  - otherwise, we get $d = 0$.

11. Using "Or"

- We want to combine these degrees of belief into a single degree of confidence that for some values $x_1, \ldots, x_n$,
  - x_1 is a possible value of the 1st input quantity,
  - and x_2 is a possible value of the 2nd quantity, ...,
  - and $y = f(x_1 \ldots, x_n)$.
- The words "for some values $x_1, \ldots, x_n$" means that the following composite property hold
  - either for one combination of real numbers $x_1, \ldots, x_n$,
  - or from another combination, etc.
- The simplest way to represent "or" is to use max.
- Thus, the desired degree of confidence $m(y)$ is equal to the maximum of the degrees corr. to different x_i:

$$m(y) = \sup_{x_1,\ldots,x_n} \min(m_1(x_1), m_2(x_2), \ldots, d(y, x_1, \ldots, x_n)).$$

12. Zadeh's Extension Principle

- $m(y) = \sup\limits_{x_1,\ldots,x_n} \min(m_1(x_1), m_2(x_2), \ldots, d(y, x_1, \ldots, x_n)).$
- We know that the maximized degree is non-zero only when $y = f(x_1 \ldots, x_n)$.
- It is therefore sufficient to only take supremum over such combinations.
- For such combinations, we can omit the term $d(y, x_1, \ldots, x_n)$ in the maximized expression.
- So, we arrive at the following formula:

  $$m(y) = \sup\{\min(m_1(x_1), m_2(x_2), \ldots) : y = f(x_1, \ldots, x_n)\}.$$

- This formula was first proposed by L. Zadeh and is thus called *Zadeh's extension principle.*
- This is the main formula that describes knowledge processing under fuzzy uncertainty.

13. Reduction to Interval Computations

- $m(y) = \sup\{\min(m_1(x_1), \ldots) : y = f(x_1, \ldots, x_n)\}$.
- Knowledge processing under fuzzy uncertainty is usually done by reducing to interval computations.
- Specifically, for each fuzzy set $m(x)$ and for each α in $(0, 1]$, we can define its α-*cut* $\mathbf{x}(\alpha) \stackrel{\text{def}}{=} \{x : m(x) \ge \alpha\}$.
- Vice versa, if we know the α-cuts for all α, we can reconstruct $m(x)$ as the largest α for which $x \in \mathbf{x}(\alpha)$.
- When $m_i(x_i)$ are fuzzy numbers, and $y = f(x_1, \ldots, x_n)$ is continuous, then for each α, we have:

$$\mathbf{y}(\alpha) = f(\mathbf{x}_1(\alpha), \ldots, \mathbf{x}_n(\alpha)).$$

- There exist many efficient algorithms and software packages for solving interval computations problems.
- So, the above reduction can help to efficiently solve the problems of fuzzy data processing as well.

14. Measurement and Estimation Inaccuracies Are Usually Small

- In many practical situations, the measurement and estimation inaccuracies Δx_i are relatively small.
- Then, we can safely ignore terms which are quadratic (or of higher order) in terms of Δx_i:

$$\Delta y = \widetilde{y} - y = f(\widetilde{x}_1, \ldots, \widetilde{x}_n) - f(\widetilde{x}_1 - \Delta x_1, \ldots, \widetilde{x}_n - \Delta x_n) = \sum_{i=1}^{n} c_i \cdot \Delta x_i, \text{ where } c_i = \frac{\partial f}{\partial x_i}.$$

- If needed, the derivative can be estimated by numerical differentiation

$$c_i \approx \frac{f(\widetilde{x}_1, \ldots, \widetilde{x}_{i-1}, \widetilde{x}_i + h, \widetilde{x}_{i+1}, \ldots, \widetilde{x}_n) - \widetilde{y}}{h}.$$

15. Estimating Accuracy of Data Processing

- The value $\Delta y = \sum\limits_{i=1}^{n} c_i \cdot \Delta x_i$ is the largest when each term is the largest, so $\Delta = \sum\limits_{i=1}^{n} |c_i| \cdot \Delta_i$.
- In the fuzzy case, the similar formula holds for the α-cuts, for every α: ${}^{\alpha}\Delta = \sum\limits_{i=1}^{n} |c_i| \cdot {}^{\alpha}\Delta_i$.
- Experts cannot describe their degrees of confidence α with too much accuracy.
- Usually, it is sufficient to consider only eleven values $\alpha = 0.0$, $\alpha = 0.1$, $\alpha = 0.2$, $\ldots$, $\alpha = 0.9$, and $\alpha = 1.0$.
- Thus, we need to apply the above formula eleven times.
- This is in line with the fact we usually divide each quantity into 7 ± 2 categories (Miller's "7 ± 2 Law").
- So, it is sufficient to have at least 9 different categories.

Part II

In Many Practical Situations, We Can Further Speed Up Fuzzy Data Processing

16. When a Further Speed-Up Is Possible?

- Sometimes, all membership functions are "of the same type": $m(z) = m_0(k \cdot z)$ for some symmetric $m_0(z)$.
- *Example:* for triangular functions,

$$m_0(z) = \max(1 - |z|, 0).$$

- In this case, $m(z) \geq \alpha$ is equivalent to $m_0(k \cdot z) \geq \alpha$, so ${}^\alpha\Delta_0 = k \cdot {}^\alpha\Delta$ and ${}^0\Delta_0 = k \cdot {}^0\Delta$.
- Thus, ${}^\alpha\Delta = f(\alpha) \cdot {}^0\Delta$, where $f(\alpha) \stackrel{\text{def}}{=} \dfrac{{}^\alpha\Delta_0}{{}^0\Delta_0}$.
- For example, for a triangular membership function, we have $f(\alpha) = 1 - \alpha$.
- So, if we know the type m_0 (hence $f(\alpha)$), and we know the 0-cut, we can compute all α-cuts as ${}^\alpha\Delta = f(\alpha) \cdot {}^0\Delta$.
- So, if $m_i(\Delta x_i)$ are of the same type, then ${}^\alpha\Delta_i = f(\alpha) \cdot {}^0\Delta_i$ for all α.

17. When a Speed-Up Is Possible (cont-d)

- We know that ${}^\alpha\Delta = \sum\limits_{i=1}^n |c_i| \cdot {}^\alpha\Delta_i$.
- For ${}^\alpha\Delta_i = f(\alpha) \cdot {}^0\Delta_i$, we get

$${}^\alpha\Delta = \sum_{i=1}^n |c_i| \cdot f(\alpha) \cdot {}^0\Delta_i.$$

- So, ${}^\alpha\Delta = f(\alpha) \cdot \left(\sum\limits_{i=1}^n |c_i| \cdot {}^0\Delta_i \right) = f(\alpha) \cdot {}^0\Delta$.
- Thus, if all $m(x)$ are of the same type $m_0(z)$, there is no need to compute ${}^\alpha\Delta$ eleven times:
  – it is sufficient to compute ${}^0\Delta$;
  – to find all other values ${}^\alpha\Delta$, we simply multiply ${}^0\Delta$ by the factors $f(\alpha)$ corresponding to $m_0(z)$.

18. A More General Case

- A more general case is:
  - when we have a list of T different types of uncertainty – i.e., types of membership functions, and
  - each approximation error Δx_i consists of $\le T$ components of the corresponding type t:

$$\Delta x_i = \sum_{t=1}^{T} \Delta x_{i,t}.$$

- For example:
  - type $t = 1$ may correspond to intervals (which are, of course, a particular case of fuzzy uncertainty),
  - type $t = 2$ may correspond to triangular membership functions, etc.

19. How This Case Is Processed Now

- *First stage:*
  - we use the known membership functions $m_{i,t}(\Delta x_{i,t})$
  - to find the memberships functions $m_i(\Delta x_i)$ that correspond to the sums Δx_i.
- *Second stage:* we use $m_i(\Delta x_i)$ to compute the desired membership function $m(\Delta y)$.
- *Problem:* on the second stage, we apply the above formula eleven times:

$$^\alpha\Delta = \sum_{i=1}^{n} |c_i| \cdot {}^\alpha\Delta_i.$$

20. The Main Idea of this Section

- We have $\Delta y = \sum\limits_{i=1}^{n} c_i \cdot \Delta x_i$, where

$$\Delta x_i = \sum_{t=1}^{T} \Delta x_{i,t}.$$

- Thus, $\Delta y = \sum\limits_{i=1}^{n} c_i \cdot \left(\sum\limits_{t=1}^{T} \Delta x_{i,t}\right)$.
- Grouping together all the terms corr. to type t, we get $\Delta y = \sum\limits_{t=1}^{T} \Delta y_t$, where $\Delta y_t \stackrel{\text{def}}{=} \sum\limits_{i=1}^{n} c_i \cdot \Delta x_{i,t}$.
- For each t, we are combining membership functions of the same type, so it is enough to compute ${}^0\Delta_t$.
- Then, we add the resulting membership functions – by adding the corresponding α-cuts.

21. Resulting Algorithm

- Let $[-{}^0\Delta_{i,t}, {}^0\Delta_{i,t}]$ be 0-cuts of the membership functions $m_{i,t}(\Delta x_{i,t})$.
- Based on these 0-cuts, we compute, for each type t, the values ${}^0\Delta = \sum\limits_{i=1}^{n} |c_i| \cdot {}^0\Delta_{i,t}$.
- Then, for $\alpha = 0$, $\alpha = 0.1$, $\ldots$, and for $\alpha = 1.0$, we compute the values ${}^\alpha\Delta_t = f_t(\alpha) \cdot {}^0\Delta_t$.
- Finally, we add up α-cuts corresponding to different types t, to come up with the expression ${}^\alpha\Delta = \sum\limits_{t=1}^{T} {}^\alpha\Delta_t$.
- *Comment.* We can combine the last two steps into a single step: ${}^\alpha\Delta = \sum\limits_{t=1}^{T} f_t(\alpha) \cdot {}^0\Delta_t$.

22. The New Algorithm Is Much Faster

- The original algorithm computed the above formula eleven times:
$$^{\alpha}\Delta = \sum_{i=1}^{n} |c_i| \cdot {}^{\alpha}\Delta_i.$$
- The new algorithm uses the corresponding formula T times, i.e., as many times as there are types.
- All the other computations are much faster, since they do not grow with the input size n.
- Thus, if the number T of different types is smaller than eleven, the new methods is much faster.
- *Example:* for $T = 2$ types (e.g., intervals and triangular $m(x)$), we get a $\dfrac{11}{2} = 5.5$ times speedup.

Part III
Type-2 Fuzzy Case: What Is Known and How to Further Speed Up Data Processing

23. Need for Type-2 Fuzzy Sets

- Fuzzy logic analyzes cases when an expert cannot describe his/her knowledge by an exact value.
- Instead, the expert describe this knowledge by using words from natural language.
- Fuzzy logic described these words in a computer understandable form – as fuzzy sets.
- In the traditional approach to fuzzy logic, the expert's degree of certainty $m_A(x)$ is a number from $[0,1]$.
- However, we consider situations when an expert cannot describe his/her knowledge by a number.
- It is not reasonable to expect that the same expert will express his/her degree of certainty by an exact number.
- It is more reasonable to expect that the expert will describe $m(x)$ also by words from natural language.

24. Type-2 Fuzzy Sets

- It is reasonable to that the expert will describe these degrees also by words from natural language.
- Thus, a natural representation of the degree $m(x)$ is not a number, but rather a new fuzzy set.
- Such situations, in which to every value x we assign a fuzzy number $m(x)$, are called *type-2* fuzzy sets.
- Type-2 fuzzy sets provide a more adequate representation of expert knowledge.
- It is thus not surprising that in comparison with the more traditional type-1 sets, such sets lead to
  - a higher quality control,
  - higher quality clustering, etc.
- If type-2 fuzzy sets are more adequate, why are not they used more?

25. The Main Obstacle to Using Type-2 Fuzzy Sets

- Main reason: transition to type-2 fuzzy sets leads to an increase in computation time.
- Indeed, to describe a traditional (type-1) membership function function, it is sufficient to describe,
  - for each value x,
  - a single number $m(x)$.
- In contrast, to describe a type-2 set,
  - for each value x,
  - we must describe the entire membership function – which needs several parameters to describe.
- We need more numbers just to store such information.
- So, we need more computational time to process all the numbers representing these sets.

26. Interval-Valued Fuzzy Sets

- In line with this reasoning:
  - the most widely used type-2 fuzzy sets are
  - the ones which require the smallest number of parameters to store.
- We are talking about *interval-valued* fuzzy numbers, in which:
  - for each x,
  - the degree of certainty $m(x)$ is an interval

$$\mathbf{m}(x) = [\underline{m}(x), \overline{m}(x)].$$

- To store each interval, we need exactly two numbers.
- This is the smallest possible increase over the single number needed to store the type-1 value $m(x)$.

27. Towards Fast Algorithms for Processing Interval-Valued Fuzzy Data (Mendel et al.)

- For interval-valued fuzzy data, we only know the interval $\mathbf{m}_i(x_i) = [\underline{m}_i(x), \overline{m}_i(x)]$ of possible values of $m_i(x_i)$.
- By applying Zadeh's extension principle to different $m_i(x_i) \in [\underline{m}_i(x), \overline{m}_i(x)]$, we get different values of

  $$m(y) = \sup\{\min(m_1(x_1), m_2(x_2), \ldots) : y = f(x_1, \ldots, x_n)\}.$$
- When the values $m_i(x_i)$ continuously change, the value $m(y)$ also continuously changes.
- We want to know the set of possible values of $m(y)$.
- So, for every y, the set $\mathbf{m}(y)$ of all possible values of $m(y)$ is an interval:

  $$\mathbf{m}(y) = [\underline{m}(y), \overline{m}(y)].$$
- Thus, to describe this set, it is sufficient, for each y, to describe the endpoints $\underline{m}(y)$ and $\overline{m}(y)$.

28. Towards Fast Algorithms for Processing Interval-Valued Fuzzy Data (cont-d)

- We want to compute the range of

$$m(y) = \sup\{\min(m_1(x_1), m_2(x_2), \ldots) : y = f(x_1, \ldots, x_n)\}.$$

- This expression is non-strictly increasing in $m_i(x_i)$, so:
  - $m(y)$ attains its smallest value when all the inputs $m_i(x_i)$ are the smallest:

$$\underline{m}(y) = \sup\{\min(\underline{m}_1(x_1), \underline{m}_2(x_2), \ldots) : y = f(x_1, \ldots, x_n)\};$$

  - $m(y)$ attains its largest value when all the inputs $m_i(x_i)$ are the largest:

$$\overline{m}(y) = \sup\{\min(\overline{m}_1(x_1), \overline{m}_2(x_2), \ldots) : y = f(x_1, \ldots, x_n)\}.$$

- So, we need to apply Zadeh's extension principle to lower and membership functions $\underline{m}_i(x_i)$ and $\overline{m}_i(x_i)$.

29. Fast Algorithms for Processing Interval-Valued Fuzzy Data

- To find $\underline{m}(y)$ (corr., $\overline{m}(y)$), we apply Zadeh's extension principle to membership f-s $\underline{m}_i(x_i)$ (corr., $\overline{m}_i(x_i)$).
- For type-1 fuzzy sets, Zadeh's extension principle can be reduced to interval computations.
- Let $\underline{\mathbf{y}}(\alpha)$ denote α-cuts for $\underline{m}(y)$, and let $\overline{\mathbf{y}}(\alpha)$ denote α-cuts for $\overline{m}(y)$.
- Then, we arrive at the following algorithm: for every $\alpha \in (0,1]$,
  - first compute

$$\underline{\mathbf{x}}_i(\alpha) \stackrel{\text{def}}{=} \{x_i : \underline{m}_i(x_i) \geq \alpha\} \text{ and } \overline{\mathbf{x}}_i(\alpha) \stackrel{\text{def}}{=} \{x_i : \overline{m}_i(x_i) \geq \alpha\};$$

  - then compute

$$\underline{\mathbf{y}}(\alpha) = f(\underline{\mathbf{x}}_1(\alpha), \ldots, \underline{\mathbf{x}}_n(\alpha)); \;\; \overline{\mathbf{y}}(\alpha) = f(\overline{\mathbf{x}}_1(\alpha), \ldots, \overline{\mathbf{x}}_n(\alpha)).$$

30. Extension to General Type-2 Fuzzy Numbers

- *Reminder:* Zadeh's extension principle

  $$m(y) = \sup\{\min(m_1(x_1), m_2(x_2), \ldots) : y = f(x_1, \ldots, x_n)\}.$$

- *General type-2 case:* $m_i(x_i)$ are fuzzy numbers, with β-cuts $(m_i(x_i))(\beta) = [\underline{(m_i(x_i))(\beta)}, \overline{(m_i(x_i))(\beta)}]$.

- Due to known relation with interval computations:

$$(m(y))(\beta) = \sup\{\min((m_1(x_1))(\beta), \ldots) : y = f(x_1, \ldots, x_n)\}.$$

- Due to monotonicity:

$$\underline{(m(y))(\beta)} = \sup\{\min(\underline{(m_1(x_1))(\beta)}, \ldots) : y = f(x_1, \ldots, x_n)\};$$

$$\overline{(m(y))(\beta)} = \sup\{\min(\overline{(m_1(x_1))(\beta)}, \ldots) : y = f(x_1, \ldots, x_n)\}.$$

- Due to known relation with interval computations:

$$\underline{\mathbf{y}}(\alpha, \beta) = f(\underline{\mathbf{x}}_1(\alpha, \beta), \ldots);\ \ \overline{\mathbf{y}}(\alpha, \beta) = f(\overline{\mathbf{x}}_1(\alpha, \beta), \ldots),\ \text{where}$$

$$\underline{\mathbf{y}}(\alpha, \beta) \stackrel{\text{def}}{=} \{y : \underline{m(y)(\beta)} \geq \alpha\},\ \ \overline{\mathbf{y}}(\alpha, \beta) \stackrel{\text{def}}{=} \{y : \overline{m(y)(\beta)} \geq \alpha\}.$$

31. Conclusions of this Section

- Type-2 fuzzy sets more adequately describe expert's opinion than the more traditional type-1 fuzzy sets.
- The use of type-2 fuzzy sets has thus led to better quality control, better quality clustering, etc.
- *Main obstacle:* the computational time of data processing increases.
- *Known result:* processing *interval-valued* fuzzy numbers can be reduced to interval computations.
- *Conclusion:* processing interval-valued fuzzy data is (almost) as fast as processing type-1 fuzzy data.
- In this talk, we showed that fast algorithms can be extended to *general* type-2 fuzzy numbers.
- This will hopefully lead to more practical applications of type-2 fuzzy sets.

Part IV
Beyond min t-Norms

32. Need to Go Beyond min t-Norm

- Usually, Zadeh's extension principle is applied to the situations in which we use the min "and"-operation

$$f_\&(a, b) = \min(a, b).$$

- However, in many practical situations, other "and"-operations more adequately describe expert reasoning.
- It is therefore desirable to consider the general case

$$m(y) = \sup\{f_\&(m_1(x_1), m_2(x_2), \ldots : u = f(x_1, \ldots, x_n)\}.$$

- We are interested in the linearized case, when

$$\Delta y = \sum_{i=1}^{n} c_i \cdot \Delta x_i.$$

- How can we speed up computations in this general case?

33. We Can Reduce This Problem to Computing the Sum of Two Fuzzy Quantities

- We know that $\Delta y = \sum\limits_{i=1}^{n} y_i$, where $y_i \stackrel{\text{def}}{=} c_i \cdot \Delta x_i$.
- Once we know the membership f-n $m_i(\Delta x_i)$, we can easily find the membership f-n for y_i as $m_i(y_i/c_i)$.
- If we know how to find the membership f-n for the sum of two fuzzy quantities, then we can:
  - find the membership f-n for $s_2 = y_1 + y_2$;
  - then, find membership f-n for
$$s_3 = s_2 + y_2 \ (= y_1 + y_2 + y_3),$$
  - etc, until we reach $s_n = y$.
- For fuzzy quantities with membership f-ns $n_1(x_1)$ and $n_2(x_2)$, the membership f-n for $y = x_1 + x_2$ is
$$n(y) = \max_{x_1} f_{\&}(n_1(x_1), n_2(y - x_1)).$$

34. Reduction to the Case of Product t-Norm

- We want to compute

$$n(y) = \max_{x_1} f_\&(n_1(x_1), n_2(y - x_1)).$$

- Some t-norms are *Archimedean*; for them:

$$f_\&(a, b) = f^{-1}(f(a) \cdot f(b)) \text{ for some function } f(x).$$

- Archimedean t-norms are universal approximators:
  - for every $\varepsilon > 0$,
  - every t-norm $f_\&$ is ε-close to some Archimedean.
- Thus, from the practical viewpoint, we can safely assume that $f_\&$ is Archimedean.
- Then, for $q_i(x_i) \stackrel{\text{def}}{=} f(n_i(x_i))$ and $q(y) \stackrel{\text{def}}{=} f(n(y))$:

$$q(y) = \max_{x_1} q_1(x_1) \cdot q_2(y - x_1).$$

35. Reduction to Informal Convolution

- We want to compute $q(y) = \max\limits_{x_1} q_1(x_1) \cdot q_2(y - x_1)$.
- For $\ell_i(x_i) \stackrel{\text{def}}{=} -\ln(q_i(x_i))$ and $\ell(y) \stackrel{\text{def}}{=} -\ln(q(y))$:
$$\ell(y) = \min_{x_1}(\ell_1(x_1) + \ell_2(y - x_1)).$$
- This operation is known as *infimal convlution*, and denoted by $\ell = \ell_1 \,\Box\, \ell_2$.
- It is known that $\ell_1 \Box \ell_2 = (\ell_1^* + \ell_2^*)^*$ for *Legendre transform*
$$\ell^*(s) \stackrel{\text{def}}{=} \sup_x (s \cdot x - \ell(x)).$$
- There exists a linear-time algorithm for computing Legendra transform.
- Thus, we can compute $\ell_1 \,\Box\, \ell_2$ in linear time.

36. Resulting Linear Time Algorithm for Fuzzy Data Processing Beyond min t-Norm

- We are given:
  - a function $f(x_1, \ldots, x_n)$;
  - n membership functions $m_1(x_1)$, ..., $m_n(x_n)$; and
  - an "and"-operation $f_\&(a, b)$.
- We want to compute a new membership function

$$m(y) = \max\{f_\&(m_1(x_1), \ldots, m_n(x_n)) : f(x_1, \ldots, x_n) = y\}.$$

- First, we represent $f_\&$ in the Acrhiemdean form $f_\&(a, b) = f^{-1}(f(a) \cdot f(b))$ for an appropriate $f(x)$.
- We thus assume that we have algorithms for computing $f(x)$ and the inverse function $f^{-1}(x)$.
- Then, for each i, we find the value $\widetilde{x}_i$ for which $m_i(x_i)$ attains its largest possible value $m_i(\widetilde{x}_i) = 1$.

37. Algorithm (cont-d)

- We then compute the values $c_0 = f(\widetilde{x}_1, \ldots, \widetilde{x}_n)$, $c_i = \dfrac{\partial f}{\partial x_i}_{|x_i=\widetilde{x}_i}$, and $a_0 = c_0 - \sum\limits_{i=1}^{n} c_i \cdot \widetilde{x}_i$.
- Then, $f(x_1, \ldots, x_n) \approx a_0 + \sum\limits_{i=1}^{n} c_i \cdot x_i$.
- After that, e compute the membership f-ns $n_1(s_1) = m_1((s_1 - a_0)/c_1)$ and $n_i(y_i) = m_i(y_i/c_i)$ for $i > 2$.
- In terms of the variables $s_1 = a_0 + c_1 \cdot x_1$ and $y_i = c_i \cdot x_i$, the desired quantity y has the form $y = s_1 + y_2 + \ldots + y_n$.
- We compute the minus logarithms of the resulting functions: $\ell_i(y_i) = -\ln(n_i(y_i))$.
- For each i, we then use the Fast Legendre Transform algorithm to compute ℓ_i^*.

38. Algorithm (final)

- Then, we add all these Legendre transforms and apply the Fast Legendre Transform once again, getting:

$$\ell = (\ell_1^* + \ldots + \ell_n^*)^*.$$

- This function $\ell(y)$ is equal to $\ell(y) = -\ln(n(y))$, so we can reconstruct $\nu(y)$ as $n(y) = \exp(-\ell(y))$.
- Finally, we can compute the desired membership function $m(u)$ as $m(y) = f^{-1}(n(y))$.

39. Conclusion for this Section

- To process fuzzy data, we need to use Zadeh's extension principle.
- In principle, this principle can be used for any t-norm.
- However, usually, it is only used for the min t-norm.
- Reason: only for this t-norm, an efficient (linear-time) algorithm for fuzzy data processing was known.
- In many practical situations, other t-norms are more adequate in describing expert's reasoning.
- We have shown that similar efficient linear-time algorithms can be designed for an arbitrary t-norm.
- Thus, it is possible to use a more adequate t-norm – and keep fuzzy data processing efficient.

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