For Quantum and Reversible Computing, Intervals Are More Appropriate Than General Sets, And Fuzzy Numbers Than General Fuzzy Sets

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1. Need for Quantum Computing

- Our current computers are very fast in comparison with what was available a few years ago.
- However, there are still computational tasks that necessitate even faster computers.
- To speed up computers, we need to squeeze in more cells and into the same volume.
- For that, we need to make cells as small as possible.
- Already, the existing cells contain a small number of molecules.
- If we decrease them further, they will contain a few molecules.
- Thus, we will need to take into account quantum effects.

2. Quantum Computing: Additional Advantages

- There are innovative algorithms specifically designed for quantum computing.
- We can decrease the time needed to find an element in an unsorted array of size n from n to $\sqrt{n}$ steps.
- We can reduce the time needed to factor large integers of n digits from exponential to polynomial in n.
- This task is needed to decode currently encoded messages.

3. Need for Reversible Computing

- One challenge in designing quantum computers is that on the quantum level, all equations are time-reversible.
- In the traditional algorithms, even the simplest "and"-operation $a, b \to a \,\&\, b$ is not reversible:
  - even if we know that $a = a \,\&\, b =$ "false",
  - we cannot uniquely reconstruct input b.
- Reversibility is also important because, according to statistical physics:
  - any irreversible process means increasing entropy,
  - and this leads to heat emission.
- Overheating is one of the reasons why we cannot pack too many processing units into the same volume.
- So, to pack more, it is desirable to reduce this heat emission – e.g., by using only reversible computations.

4. Need to Take Uncertainty into Account

- We use computers mostly to process data.
- When processing data, we need to take into account that data comes from measurements.
- Measurements are never absolutely accurate.
- The measurement result $\widetilde{x}$ is, in general, different from the actual value x of the corresponding quantity.
- It is therefore necessary to take this uncertainty into account when processing data.

5. Need for Interval Uncertainty

- In many real life situations:
  - the only information that we have about the measurement error $\Delta x \stackrel{\text{def}}{=} \widetilde{x} - x$ is
  - the upper bound Δ on its absolute value:

$$|\Delta x| \leq \Delta.$$

- Once we have a measurement result $\widetilde{x}$, then:
  - the only information that we can conclude about the actual value x is that
  - this value is somewhere in the interval $[\widetilde{x} - \Delta, \widetilde{x} + \Delta]$.
- Such interval uncertainty indeed appears in many practical applications.

6. Data Processing under Interval Uncertainty

- In a data processing algorithm:
  - we take several inputs $x_1, \ldots, x_n$, and
  - we apply an appropriate algorithm to generate the result y depending on these inputs.
- Let us denote this dependence by $f(x_1, \ldots, x_n)$.
- For each input i, we only know the interval $X_i = [\widetilde{x}_i - \Delta_i, \widetilde{x}_i + \Delta_i]$ of possible values of x_i.
- Then, the only information that we can have about y is that y belongs to the set

$$Y = f(X_1, \ldots, X_n) \stackrel{\text{def}}{=}$$

$$\{f(x_1, \ldots, x_n) : x_1 \in X_1, \ldots, x_n \in X_n\}.$$

- When the sets X_i are intervals and the function $f(x_1, \ldots, x_n)$ is continuous, the resulting set Y is also an interval.

7. Interval Uncertainty (cont-d)

- In most practical situations, the measurement errors are relatively small.
- So, we can expand the function $f(x_1, \ldots, x_n)$ in Taylor series and retain only linear terms.
- Then, we get

$$f(x_1, \ldots, x_n) = f(\widetilde{x}_1 - \Delta x_1, \ldots, \widetilde{x}_n - \Delta x_n) \approx$$

$$\widetilde{y} - \sum_{i=1}^{n} c_i \cdot \Delta x_i, \quad \widetilde{y} \stackrel{\text{def}}{=} f(\widetilde{x}_1, \ldots, \widetilde{x}_n), \quad c_i \stackrel{\text{def}}{=} \frac{\partial f}{\partial x_i}_{|x_i = \widetilde{x}_i}.$$

- In other words, $f(x_1, \ldots, x_n)$ becomes a linear function:

$$f(x_1, \ldots, x_n) = c_0 + \sum_{i=1}^{n} c_i \cdot x_n, \quad c_0 \stackrel{\text{def}}{=} \widetilde{y} - \sum_{i=1}^{n} c_i \cdot \widetilde{x}_i.$$

- In other words, data processing can be, in effect, reduced to multiplication by a constant c_i and addition.

8. When Is This Data Processing Reversible?

- Multiplication by a constant is always reversible.
- Indeed, if we know the interval $Y = c \cdot X$, then, we can reconstruct X as $X = c^{-1} \cdot Y$.
- Addition $y = x_1 + x_2$ is also reversible.
- Indeed, if we know that $x_1 \in [\underline{x}_1, \overline{x}_1]$ and $x_2 \in [\underline{x}_1, \overline{x}_2]$, then $Y = [\underline{y}, \overline{y}]$ has the form

$$Y = [\underline{x}_1 + \underline{x}_2, \overline{x}_1 + \overline{x}_2].$$

- If we know $Y = [\underline{y}, \overline{y}]$ and $X_1 = [\underline{x}_1, \overline{x}_1]$, then we can reconstruct $X_2 = [\underline{x}_2, \overline{x}_2]$ as

$$\underline{x}_2 = \underline{y} - \underline{x}_1 \text{ and } \overline{x}_2 = \overline{y} - \overline{x}_1.$$

9. From Interval Uncertainty to a More General Set Uncertainty

- In some cases:
  - in addition to knowing that values of x are within a certain interval $[\underline{x}, \overline{x}]$,
  - we also know that some values from this interval are not possible.
- In this case, the set X of possible values of x is different from an interval.
- No matter how crude the measurements are, there is always an upper bound Δ on the measurement error.
- Thus, all possible values of x are in the interval

$$[\widetilde{x} - \Delta, \widetilde{x} + \Delta].$$

- Thus, the set X is bounded.

10. Set Uncertainty (cont-d)

- In general, we can safely assume that the set X is closed.
- Indeed, suppose that x_0 is a limit point of the set.
- Then, for every $\varepsilon > 0$, there are elements $x \in X$ is any ε-neighborhood $(x_0 - \varepsilon, x_0 + \varepsilon)$ of this value x_0.
- This means that:
  - no matter how accurately we measure the corresponding value,
  - we will not be able to distinguish between the limit value x_0 and a sufficient close value $x \in X$.
- It is therefore reasonable to simply assume that x_0 is possible.
- Thus, we conclude that the set of possible values of x contains all its limit points, i.e., is closed.

11. Data Processing under Set Uncertainty

- Assume that we know the set X_1 of possible values of x_1, and we know the set X_2 of possible values of x_2.
- Then the set $Y \stackrel{\text{def}}{=} X_1 + X_2$ of possible values of the sum $y = x_1 + x_2$ is equal to

$$Y = \{x_1 + x_2 : x_1 \in X_1 \text{ and } x_2 \in X_2\}.$$

- If we add any non-interval bounded closed set S to the class of all intervals, additions stops being reversible.
- For $\underline{S} \stackrel{\text{def}}{=} \inf\{x : x \in S\}$ and $\overline{S} \stackrel{\text{def}}{=} \sup\{x : x \in S\}$, we have

$$[\underline{S}, \overline{S}] + [\underline{S}, \overline{S}] = [\underline{S}, \overline{S}] + S (= [2\underline{S}, 2\overline{S}]).$$

- However, $[\underline{S}, \overline{S}] \neq S$.

12. Case of Fuzzy Uncertainty

- In many real-life situations:
  - in addition to the guaranteed upper bound Δ on the absolute value of the measurement error,
  - with some degree of certainty β, measurement errors can be bounded by a smaller bound $\Delta(\beta) < \Delta$.
- As a result:
  - in addition to the interval $[\widetilde{x} - \Delta, \widetilde{x} + \Delta]$ that is guaranteed to contain x with 100% confidence,
  - we have several narrower intervals $[\widetilde{x} - \Delta(\beta), \widetilde{x} + \Delta(\beta)]$ that contain x with confidence β.
- In other words, we have a nested family of intervals corresponding to different values β.
- The larger the β (i.e., the higher the desired confidence), the wider the interval.

13. Case of Fuzzy Uncertainty (cont-d)

- Such a family of nested interval is, in effect, an equivalent way of representing a fuzzy number.
- If instead of intervals, we have more general sets $S(\beta)$, then we have a *fuzzy set*.
- The sets $S(\beta)$ are known as α-*cuts* of the fuzzy set, where $\alpha \stackrel{\text{def}}{=} 1 - \beta$.
- For such fuzzy sets, we can define operations layer-by-layer:
  - for each β (i.e., equivalently, for each α),
  - we process all the sets (or intervals) corresponding to this value β.

14. Case of Fuzzy Uncertainty (cont-d)

- Fuzzy numbers correspond to intervals, and general fuzzy sets to general sets.
- So, we conclude that addition is only reversible for fuzzy numbers.
- If we add any fuzzy set which is not a fuzzy number to fuzzy numbers, addition stops being reversible.

15. Intervals are Ubiquitous

- We showed that intervals (and fuzzy numbers) are preferable: they lead to reversible data processing.
- Interestingly, intervals (and fuzzy numbers) are indeed ubiquitous.
- They occur much much more frequently in practice as descriptions of uncertainty than any other sets.
- Why is that?

16. A Possible Explanation: Main Idea

- Let us recall why normal (Gaussian) distributions are ubiquitous.
- The usual explanation is that usually, there are many different independent sources of measurement error.
- As a result, the measurement error is a sum of a large number of small independent random variables.
- In the limit, when the number of terms increases, the distribution of the sum tends to normal.
- This is known as the Central Limit Theorem.
- This means that when the number of components is large, the corresponding distribution is close to normal.
- Thus, from the practical viewpoint, we can safely consider the distribution to be normal.
- In non-probabilistic case, the situation is similar.

17. Main Idea (cont-d)

- The measurement error is the sum of a large number n of small independent error components:

$$\Delta x = \Delta x^{(1)} + \Delta x^{(2)} + \ldots + \Delta x^{(n)}.$$

- Let us assume that for each of the components $\Delta x^{(k)}$, we know the set $X^{(k)}$ of possible values.
- Then the set S of possible values of their sum is equal to the sum of these sets:

$$X = X^{(1)} + \ldots + X^{(n)} =$$
$$\{\Delta x^{(1)} + \Delta x^{(2)} + \ldots + \Delta x^{(n)} :$$
$$\Delta x^{(1)} \in X^{(1)}, \ldots, \Delta x^{(n)} \in X^{(n)}\}.$$

- It can be shown that, when n increases, the resulting set X also tends to an interval.

18. Need for a More Detailed Explanation

- The limit closeness is good.
- However, in practice, it is desirable to know exactly how close is the resulting set X to an interval.
- For every positive real number $\varepsilon > 0$, two points a and b are *ε-close* is $|a - b| \le \varepsilon$.
- It is therefore reasonable to say that the sets A and B are *ε-close* if:
  - every point $a \in A$ is ε-close to some point $b \in B$, and
  - every point $b \in B$ is ε-close to some point $a \in A$.
- The smallest value ε with this property is known as the *Hausdorff distance* $d_H(A, B)$ between the two sets.

19. How to Measure Smallness of a Set

- The size of a set A can be naturally measured by its *diameter* $\text{diam}(A)$.
- The diameter is the largest possible distance $d(a, a')$ between the two points a, a' from this set.
- For bounded closed subsets A of a real line, the diameter is equal to $\text{diam}(A) = \sup A - \inf A$.

20. Our Main Result

- *If* $\operatorname{diam}(A_i) \le \varepsilon$ *for all* $i = 1, \ldots, n$, *then for* $A = A_1 + \ldots + A_n$ *and for some interval* I*:*

$$d_H(A, I) \le \varepsilon/2.$$

- This bound cannot be improved, as shown by the following auxiliary result.
- *For every* n, *there exist closed bounded sets* $A_1, \ldots, A_n$ *for which* $\operatorname{diam}(A_i) \le \varepsilon$ *for all* i, *and for which*

$$d_H(A, I) \ge \varepsilon/2 \text{ for all } I.$$

21. Proof of the Main Result

- Let us show that the desired inequality holds from the interval $[\underline{a}, \overline{a}]$, where:
  - $\underline{a} \stackrel{\text{def}}{=} \underline{a}_1 + \ldots + \underline{a}_n$, where $\underline{a}_i \stackrel{\text{def}}{=} \inf A_i$, and
  - $\overline{a} \stackrel{\text{def}}{=} \overline{a}_1 + \ldots + \overline{a}_n$, where $\overline{a}_i \stackrel{\text{def}}{=} \sup A_i$.
- To prove the desired inequality, we need to show that:
  - every point $a \in A$ is $(\varepsilon/2)$-close to some point from the interval $I = [\underline{a}, \overline{a}]$, and
  - vice versa, that every point b from the interval $I = [\underline{a}, \overline{a}]$ is $(\varepsilon/2)$-close to some point from the sum A.
- Let us first prove that every point $a \in A$ is $(\varepsilon/2)$-close to some point from the interval $I = [\underline{a}, \overline{a}]$.
- Indeed, by definition of A, every point $a \in A$ has the form $a = a_1 + \ldots + a_n$, where $a_i \in A_i$ for all i.

22. Proof of the Main Result (cont-d)

- Every point $a_i \in A_i$ is bounded by this set's inf and sup: $\underline{a}_i = \inf A_i \le a_i \le \sup A_i \le \overline{a}_i$.
- Let us add up n such inequalities, and take into account that:
  - $\underline{a} = \underline{a}_1 + \ldots + \underline{a}_n$,
  - $a = a_1 + \ldots + a_n$, and
  - $\overline{a} = \overline{a}_1 + \ldots + \overline{a}_n$.
- We can then conclude that $\underline{a} \le a \le \overline{a}$, i.e., that the value a actually itself belongs to the interval I.
- So, we can take $b = a$, and get $|a - b| = 0 \le \varepsilon/2$.
- Let us prove that, vice versa, every point b from the interval I is $(\varepsilon/2)$-close to some point $a \in A$.
- Indeed, since all A_i are closed sets, they contain their limit points $\underline{a}_i = \inf A_i \in A_i$.

23. Proof of the Main Result (cont-d)

- Thus, $\underline{a} = \underline{a}_1 + \ldots + \underline{a}_n \in A$.
- Since $b \in I$, we have $b \geq \underline{a}$, so b is larger than or equal to some point $a \in A$.
- Let us define $a_0 = \sup\{a \in A : a \leq b\}$.
- Since all A_i are closed sets, the sum A of these sets is also closed.
- So, a_0, as a limit of elements from A, also belongs to A.
- In the limit, from $a \leq b$, we conclude that $a_0 \leq b$.
- If $a_0 = \overline{a}$, then, from the fact that $a_0 \leq b \leq \overline{a}$, we conclude that $b = a_0 = \overline{a}$ and thus, $|a_0 - b| = 0 \leq \varepsilon/2$.
- Let us now consider the remaining case when

$$a_0 < \overline{a} = \overline{a}_1 + \ldots + \overline{a}_n.$$

24. Proof of the Main Result (cont-d)

- Since the point a_0 is in A, it means that

$$a_0 = a_1 + \ldots + a_n \text{ for some } a_i \in A_i.$$

- For each i, we have $a_i \le \sup A_i = \overline{a}_i$.
- The inequality $a_0 < \overline{a}$ implies that we cannot have $a_i = \overline{a}_i$ for all i: otherwise, we would have

$$a_0 = a_1 + \ldots + a_n = \overline{a}_1 + \ldots + \overline{a}_n = \overline{a}.$$

- Thus, there exists an i for which $a_i < \overline{a}_i$.
- Let us denote one such index by i_0; then $a_{i_0} < \overline{a}_{i_0}$.
- Let us now consider a new point $\overline{a}_0 \in A$ in forming which we replace a_{i_0} with $\overline{a}_{i_0}$:

$$\overline{a}_0 = a_1 + \ldots + a_{i_0-1} + \overline{a}_{i_0} + a_{i_0+1} + \ldots + a_n.$$

- Here, we have $\overline{a}_0 - a_0 = \overline{a}_{i_0} - a_{i_0}$.

25. Proof of the Main Result (cont-d)

- Thus, by the definition of the diameter, this difference is smaller than or equal to the diameter $\text{diam}(A_{i_0})$.
- This diameter is $\le \varepsilon$; thus, $|\overline{a}_0 - a_0| \le \varepsilon$.
- Since a_0 is the largest point from A which is $\le b$, and $\overline{a}_0 > a_0$, we conclude that $a_0 \not\le b$, i.e., that $b < \overline{a}_0$.
- So, we have $a_0 \le b < \overline{a}_0$.
- The sum of the distances $|b - a_0|$ and $|b - \overline{a}_0|$ is equal to $|\overline{a}_0 - a_0|$ and is, thus, smaller than or equal to ε:

$$|b - a_0| + |b - \overline{a}_0| \le \varepsilon.$$

- So, at least one of these distances must be $\le \varepsilon/2$ (if they were both $> \varepsilon/2$, their sum would be $> \varepsilon$).
- In each of these two cases, we have a point from A (a_0 or $\overline{a}_0$) which is $(\varepsilon/2)$-close to $b \in I$. Q.E.D.

26. Proof of Auxiliary Result

- Let us take $A_1 = \ldots = A_n = \{0, \varepsilon\}$.
- Then, as one can easily see,

$$A = A_1 + \ldots + A_n = \{0, \varepsilon, 2 \cdot \varepsilon, \ldots, n \cdot \varepsilon\}.$$

- Let us show, by reduction to a contradiction, that we cannot have $d_H(A, I) < \varepsilon/2$ for any interval I.
- Indeed, suppose that such an interval exists.
- Then, by definition of the Hausdorff distance, for the point $0 \in A$, there exists a point $b_1 \in I$ for which

$$|b_1 - 0| = |b_1| \le d_H(A, I).$$

- Then, since $b_1 \le |b_1|$, we have $b_1 \le d_H(A, I)$.
- Since $d_H(A, I) < \varepsilon/2$, we thus have $b_1 < \varepsilon/2$.
- Similarly, for the point $\varepsilon \in A$, there exists a point $b_2 \in I$ for which $|\varepsilon - b_2| \le d_H(A, I)$.

27. Proof of Auxiliary Result (cont-d)

- Thus, $\varepsilon - b_2 \le d_H(A, I)$ and $\varepsilon - d_H(A, I) \le b_2$.
- Since $d_H(A, I) < \varepsilon/2$, we thus have $b_2 > \varepsilon - \varepsilon/2 = \varepsilon/2$.
- Since I contains two points $b_1 < \varepsilon/2$ and $b_2 > \varepsilon/2$, it contains all the points in between, including $b = \varepsilon/2$.
- However, for this point $b \in I$, the closest points from A are the points 0 and ε.
- For both of them, the distance to $b = \varepsilon/2$ is equal to $\varepsilon/2$ and is, thus, larger than $d_H(A, I)$.
- This contradicts to the definition of Hausdorff distance.
- Indeed, by this definition, every $b \in I$ is $d_H(A, I)$-close to some point from A.
- This contradiction proves that the inequality $d_H(A, I) < \varepsilon/2$ is impossible. So, $d_H(A, I) \ge \varepsilon/2$. Q.E.D.

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