Physics' Need for Interval Uncertainty and How It Explains Why Physical Space Is (at Least) 3-Dimensional

Olga Kosheleva and Vladik Kreinovich
University of Texas at El Paso
500 W. University
El Paso TX 79968, USA
olgak@utep.edu, vladik@utep.edu

1. Divergence Problem in Physics: A Brief Reminder

- The mass m of an electron consists of two parts $m = m_0 + m_f$:
  - the mass m_0 of the particle itself, and
  - the mass $m_f = \dfrac{E_f}{c^2}$ equivalent to the energy E_f of its electric field.
- According to special relativity, an elementary particle, i.e., a particle that cannot be further subdivided, it is a point-wise object.
- Otherwise, different spatial locations within the same particle would be, in general, different, since they cannot instantaneously interact.
- For a point-wise particle, the electric field at a distance r from the particle is proportional to r^{-2}: $E = c_E \cdot r^{-2}$.
- The energy density ρ of the field is proportional to E^2, thus, $\rho = c_\rho \cdot r^{-4}$.
- E_f can be obtained by integrating $\rho(x)$ over the 3-D space:

$$E_f = \int \rho(x)\, d^3x = \int c_\rho \cdot r^{-4}\, d^3x = \int_{r=0}^{r=\infty} c_\rho \cdot r^{-4} \cdot 4\pi \cdot r^2\, dr =$$

$$c_f \cdot \int_0^\infty \frac{dr}{r^2} = c_f \cdot \left.\frac{1}{r}\right|_{r=0}^{r=\infty} = \infty.$$

2. Divergence Is Caused Only by Scale-Invariance

- Can we solve this problem by changing Coulomb's law?
- The standard formulas for electromagnetic interactions (and for many other interactions) do not have a preferred unit of length.
- As a result, formulas should not change if we simply change the unit of length – e.g., replace meters with feet.
- If we replace the original unit with a unit which is λ times smaller, then all the numerical values of distance are multiplied by λ: $r \to r' \stackrel{\text{def}}{=} \lambda \cdot r$.
- x and y may be related, so when we re-scale x, we may also need to re-scale y, to $y' = c(\lambda) \cdot y$.
- We can now require that when $y = f(x)$, then $y' = f(x')$.
- Substituting the expressions for x' and y' into this formula, we conclude that $f(\lambda \cdot x) = c(\lambda) \cdot f(x)$.
- For continuous (even for measurable) functions $f(x)$ this equality implies that $f(x) = A \cdot x^c$ for some A and c, so $\rho(r) = A \cdot r^c$.

3. Divergence Caused by Scale-Invariance (cont-d)

- In an m-dimensional space, for a spherically symmetric function $\rho(r)$ (i.e., a function depending only on r), we thus have

$$E_f = \int \rho(r)\, d^m x = A \cdot \int r^c\, d^m x = A \cdot \text{const} \cdot \int_0^\infty r^c \cdot r^{m-1}\, dr = \text{const} \cdot \left. \frac{r^{c+m}}{c+m} \right|_0^\infty$$

- This expression is always infinite:
  - when $c + m > 0$, it is infinite for $r = \infty$;
  - when $c + m < 0$, the above expression is infinite for $r = 0$;
  - finally, when $c + m = 0$, the integral of $r^{c+m-1} = r^{-1}$ is equal to $\ln(r)$, and is, therefore, infinite both at $r = 0$ and at $r = \infty$.
- Thus, the divergence problem indeed follows from scale-invariance.
- It does not depend on what exactly is the dependence on the field on the distance or on what is the dimension of the physical space.

4. How the Divergence Problem Is Solved Now: Renormalization

- Crudely speaking, we take $m_0 = -\infty$, so that the sum of minus infinity m_0 and plus infinity $m_f = \dfrac{E_f}{c^2}$ is finite.
- To implement this idea, instead of considering point-wise particles, we consider particles of a finite radius ε.
- In this case, the overall energy $E_f(\varepsilon)$ is finite. We then take $m_0(\varepsilon) = m - \dfrac{E_f(\varepsilon)}{c^2}$, where m is the empirically observed electron's mass.
- In the limit $\varepsilon \to 0$, the proper mass $m_0(\varepsilon)$ tends to $-\infty$, the energy $E_f(\varepsilon)$ tends to $+\infty$, but their sum remains equal to m.
- From the mathematical viewpoint, renormalization solves the divergence problem.
- However, from the physical viewpoint, renormalization looks like a trick.

5. Alternative Approaches: Main Idea and Limitations

- Many physical interactions – such as strong ones – correspond to exchanging quanta of finite rest mass.
- For such interactions, the force exponentially decreases with distance, and this decrease makes the corresponding integrals finite.
- So, all $r = \infty$ problems will disappear if we assume that, e.g., photons – quanta of electromagnetic interactions – also have a non-zero rest mass.
- To avoid infinities at $r = 0$, a natural idea is to modify the formulas corresponding to small r.
- This can be done, e.g., by considering physical theories in which there is a quantum of length ("elementary length").
- The main problem with these approaches is that they have not been experimentally confirmed.
- Thus, from the physical viewpoint, these alternative approaches also sounds like mathematical tricks.
- We need a more physically meaningful way to avoid infinities.

6. Main Idea: Interval Uncertainty

- We implicitly assumed that each physical quantity can be, in principle, measured with any possible accuracy.
- Thus, we assumed that each quantity can be characterized by an exact number.
- In practice, measurements also have some imprecision.
- It is reasonable to consider the case when we cannot perform measurements beyond a certain level of accuracy.
- In this case, we will never know the exact value of the corresponding quantity; we will only know the interval containing this quantity.
- Let us see how this possibility affects the divergence.

7. Interval Uncertainty Helps Avoid Divergence

- Let us assume the distance r can only be measured with accuracy r_0 (and is, thus, defined only with this accuracy).
- In this case, when the measured distance is r, the actual distance can take all possible values:
  - from $r - r_0$ (to be more precise, from $\max(r - r_0, 0)$ since the distance is always non-negative)
  - to $r + r_0$.
- The expression $\rho(x) = c_\rho \cdot r^{-4}$ for the dependence of energy density ρ on the distance r is decreasing.
- Thus, the density $\rho(x)$ of the energy field can take any value from $\underline{\rho}(x) = c_\rho \cdot (r + r_0)^{-4}$ to $\overline{\rho}(x) = c_\rho \cdot (\max(0, r - r_0))^{-4}$.
- Therefore, the total energy E_f of the particle's electric field lies between

  $\underline{E}_f = \int c_\rho \cdot (r + r_0)^{-4}\, d^3x = 4\pi \cdot c_\rho \cdot \int_0^\infty r^2 \cdot (r + r_0)^{-4}\, dr$ and $\overline{E}_f = \infty$.

8. Interval Uncertainty Helps Avoid Divergence (cont-d)

- The lower bound $\underline{E}_f$ is finite, since the lower density is bounded by a constant in the vicinity of $r = 0$.
- Thus, the fact that we have an infinite upper bound does not imply that the energy is infinite.
- The energy can be finite, as long as it is larger than or equal to $\underline{E}_f$.
- In short, in the presence of interval uncertainty, there is no divergence.
- It is easy to see that the divergence at $r = 0$ disappears for all possible power law dependencies $\rho(x) \sim r^c$ and for all possible spatial dimensions.
- In addition to a fixed-accuracy interval uncertainty, we can also consider a more realistic model, in which we have:
  - different accuracies
  - with different degree of confidence.
- As a result, for each quantity, instead of a single interval, we have a nested family of intervals corresponding to different degrees of confidence.
- This is, in effect, a fuzzy set.

9. Interval-Valued Quantities Naturally Appear

- According to quantum physics, all quantities can be determined only with some probabilities.
- From this viewpoint, the most important quantity is the probability.
- In mathematical terms, it is a probability measure on the set of all possible events.
- Even for the usual Lebesgue measure (length, area, volume, etc.) some sets S are not measurable.
- The lower probability $\underline{P}(S) = \sup\{P(A) : A \subseteq S\}$ differs from the upper probability $\overline{P}(S) = \inf\{P(A) : S \subseteq A\}$.
- In such cases, we can say that the measure of the set S is the interval

$$[\underline{P}(S), \overline{P}(S)].$$

- So, it is possible (and natural) to have interval-valued probabilities.
- Thus, it is natural to have interval-type values for all other quantities, such as the expected values, moments, etc.

10. In 2-D Case, It Is possible to Avoid Intervals Altogether

- Interval-valued probabilities are not necessarily appearing in 2-D space.
- Indeed, it is possible to extend Lebesgue measure to a shift- and rotation-invariant finitely-additive measure defined on all possible 2-D sets.
- In 3-D space:
  - we can decompose a unit ball (= filled sphere) into finitely many pieces, and then
  - shift and rotate these pieces so that these shifted-and-rotated pieces form two balls identical to the original one.
- This counterintuitive possibility is known as *Banach-Tarski paradox*.
- This result shows that in the 3-D case, it is not possible to extend Lebesgue to shift- and rotation-invariant finitely additive measure.
- A similar construction is possible for all higher dimensions.

11. This Explains Why the Physical Space Is at ≥3-Dimensional

- As we have mentioned earlier, to avoid physically meaningless divergences, it is necessary to have interval uncertainty.
- In 1-D and 2-D cases, it is possible to avoid interval uncertainty – and thus, get divergence.
- Starting with dimension 3, however, interval values are inevitable and thus, divergence is not possible.
- This explains why the physical space is at least 3-dimensional – with the usual 3-D physical space being the simplest space with this property.

14. Acknowledgments

This work was partially supported by the US National Science Foundation via grant HRD-1242122 (Cyber-ShARE Center of Excellence).