Interval-Valued and Set-Valued Extensions of Discrete Fuzzy Logics, Belnap Logic, and Color Optical Computing

Victor L. Timchenko[1], Yury P. Kondratenko[2,3], Vladik Kreinovich[4], and Olga Kosheleva[5]

[1]Admiral Makarov National Univ. of Shipbuilding, Mykolaiv 054025, Ukraine
vl.timchenko58@gmail.com

[2]Petro Mohyla Black Sea National Univ., Mykolaiv 054003, Ukraine
y_kondrat2002@yahoo.com

[3]Institute of Artificial Intelligence Problems,
Ministry of Education and Science of Ukraine (MES) and
the National Academy of Sciences of Ukraine (NAS)
Kyiv 01001, Ukraine

[4]Department of Computer Science, University of Texas at El Paso
El Paso, Texas 79968, USA, vladik@utep.edu

1. Color optical computing representation of fuzzy degrees

- It has been recently shown that:
  - in some practical applications of fuzzy logic – e.g., in ship navigation near a harbor,
  - it is convenient to represent different fuzzy degrees by colors,
  - namely, by combinations of the three pure basic colors: red, green, and blue.
- these papers use $2^3 = 8$ combinations of pure colors, where each of the three basic colors is either present or not present:
  - *black* corresponding to no colors at all,
  - *white* corresponding to the presence of all three basic colors,
  - three pure colors corresponding to the case when only one of the three basic colors is present, and
  - three combinations of two basic colors.
- Natural question: why is this representation efficient?

2. What we do in this talk

- In this talk, we explain the empirical success of color optical computing representation.
- We show that the main ideas behind fuzzy logic naturally lead to this representation.
- Namely, we show:
  - that it is reasonable to consider discrete fuzzy logics,
  - that it is reasonable to consider interval-valued and set-valued extensions of these logics, and
  - that a set-valued extension of the 3-valued logic is naturally equivalent to the use of combinations of pure colors.
- We also show that the set-valued extensions of discrete fuzzy logics are related to the formalism of Belnap's logic.
- That logic allows parts of the knowledge base to be inconsistent.

3. Need for discrete fuzzy logic

- One of the main ideas behind fuzzy logic is to assign:
  - to each imprecise natural-language statement such as "John is tall",
  - a degree describing to what extent this statement is true – e.g., to what extent John is tall.
- In the original fuzzy logic, these degrees were represented by numbers from the interval $[0, 1]$.
- From the mathematical viewpoint, this interval contains infinitely many numbers.
- When the numbers are significantly different, they represent different degrees of certainty.
- However, when the two numbers are very close, we cannot distinguish the corresponding degrees.

4. Need for discrete fuzzy logic (cont-d)

- For example, hardly anyone can distinguish between degrees 0.80 and 0.81.
- In general, according to psychological experiments, we can meaningfully distinguish at most 7 ± 2 different degrees:
  - some of us can only distinguish $7 - 2 = 5$ different degrees,
  - some can distinguish $7 + 2 = 9$ different degrees.
- In other words, in practice, we use, in effect, a discrete set of fuzzy degrees.

5. Fuzzy degrees come with uncertainty

- In the ideal case:
  - we have a single perfect expert who selects a single degree, and
  - experts are perfect – in the sense that other experts would assign the exact same degree.
- In practice, the situation is more complicated.
- First, an expert can be unsure what exact degree to assign.
- At best, the expert can provide a lower bound a and an upper bound b for this degree.
- Similarly, when estimating the height of a person entering the room, the expert will not produce an exact value but rather a range of values.
- In this case, possible degrees form an *interval* $[a, b] \stackrel{\text{def}}{=} \{x : a \le x \le b\}$.

6. Fuzzy degrees come with uncertainty (cont-d)

- Second, even if an expert produces an exact degree, other experts may produce different degrees.
- In this case, to describe uncertainty, it is reasonable to list all these degrees.
- So, we should produce the *set* of experts' estimates.
- This extension of fuzzy logic is known as *hesitant* fuzzy logic.
- In the following slides, we will analyze such interval-valued and set-valued versions of the simplest discrete fuzzy logics.
- We will show that this analysis indeed naturally leads to color optical computing.
- By the way:
  - Following this line of reasoning, it is also possible to have several experts producing intervals.
  - This option may be worth exploring.

7. 2-valued logic

- In general, a discrete fuzzy logic is a finite subset of the interval $[0, 1]$ that contains both 0 (“false”) and 1 (”true”).
- From this viewpoint, the simplest case is:
  - when this subset contains only 0 and 1,
  - i.e., when we have a usual 2-valued logic.

8. Interval-values extension of 2-valued logic

- In a logic consisting of two elements $0 < 1$, there are exactly three possible intervals:
  - two degenerate intervals $[0, 0] = \{0\}$ and $[1, 1] = \{1\}$ consisting of a single original value, and
  - a non-degenerate interval $[0, 1] = \{0, 1\}$ containing both values.
- The above interpretation of interval-valued extensions provides the following explanation for the new truth value $[0, 1]$.
- This truth value corresponds to the case when we do not know whether the statement is true or false.
- In other words, it corresponds to uncertainty.
- Thus, we get a usual 3-valued logic with three possible truth values: true, false, and uncertain.
- These values can be naturally described as 1, 0, and an intermediate value 0.5.

9. Set-valued extension of 2-valued logic

- In a 2-valued logic with the set of truth values $\{0, 1\}$, there are four subsets:
  - two 1-elements subsets $\{0\}$ and $\{1\}$;
  - the original set $\{0, 1\}$, and
  - the empty set $\emptyset$.
- The above interpretation of set-valued extensions provides the following interpretation of these four subsets.
- The set $\{0\}$ means that all experts agree that the statement is false.
- The set $\{1\}$ means that all experts agree that the statement is true.
- The set $\{0, 1\}$ means that some experts believe that the statement is true, while some other experts believe that the statement is false.
- Finally, the empty set means that no experts have any opinion about this statement.

10. Set-valued extension of 2-valued logic (cont-d)

- Here, both the set $\{0, 1\}$ and the empty set correspond to uncertainty.
- However, there is a difference between the two cases.
- The empty set means, in effect, that we know nothing about the statement.
- In contrast, the set $\{0, 1\}$ means, in effect, that we have:
  - some arguments in favor of the given statement, and
  - some arguments against this statement.

11. How is this related to interval-valued fuzzy techniques

- The need to distinguish between these two types of uncertainty is often emphasized as the need to go:
  - from the traditional fuzzy logic
  - to its interval-valued version.
- Indeed, in the traditional fuzzy logic, the same value 0.5 can mean two different things:
  - it can mean that we know nothing about the given statement, and
  - it can also mean that we have as many arguments in favor of this statement as against it.

12. How is this related to interval-valued fuzzy techniques (cont-d)

- In the interval-valued case:
  - the first situation – when we know nothing, the statement can be false or true,
  - is naturally described by the interval $[0, 1]$ containing all possible truth values.
- For the second situation, a value 0.5 – corresponding to the degenerate (1-point) interval $[0.5, 0.5]$ seems to be a better match.

13. How is this related to Belnap logic

- The above four truth values have been analyzed in a non-fuzzy context, under the name of Belnap logic.
- In this context, instead of expert opinions about the truth of a statement, we consider the actual validity of this statement.
- In this interpretation, the set $\{0, 1\}$ corresponds to inconsistency.
- It means that our knowledge base mistakenly contains both:
  - the information that this statement is true and
  - the information that this same statement is false.

14. How is this related to Belnap logic (cont-d)

- The need to consider this logic was caused by the fact that in the usual 2-valued logic:
  - once we have a single contradiction,
  - we can conclude that all statements are true,
  - and that all statements are false.
- So, if we use the usual logic:
  - one wrong statement added to the database – e.g., that the train leaves at 1 pm and that this same train leaves at 1.01 pm
  - would make the whole knowledge base useless.

15. 3-valued logic

- After the simplest 2-valued logic, the next simplest is 3-valued logic.
- In this logic, we add, to the usual 0 (“false”) and 1 (”true”), and additional intermediate degree corresponding to uncertainty.
- For simplicity, let us denote this degree by 0.5.

16. Interval-valued extension of 3-valued logic

- For this logic, with 3 truth values $0 < 0.5 < 1$, there are six possible intervals.
- The degenerate interval $[0, 0] = \{0\}$ means that the expert believes that the given statement is false.
- The degenerate interval $[1, 1] = \{1\}$ means that the expert believes that the given statement is true.
- The degenerate interval $[0.5, 0.5] = \{0.5\}$ means that the expert is uncertain.
- The interval $[0, 0.5] = \{0, 0.5\}$ means the expert is uncertain but is leaning towards "false".
- The interval $[0.5, 1] = \{0.5, 1\}$ means the expert is uncertain but is leaning towards "true".
- The interval $[0, 1] = \{0, 0.5, 1\}$ means that the expert is uncertain, but has some arguments in favor and against the given statement.

17. Interval-valued extension of 3-valued logic (cont-d)

- In the 2-valued case, the interval extension did not allow us to distinguish between two different situations:
  - not having any information about a statement and
  - having arguments for and argument against the statement.
- To distinguish between these two cases, we had to consider set-valued extension of the 2-valued logic.
- Interesting, in the 3-valued case, already the interval extension enables us to distinguish between these two situations.

18. Set-valued extension of 3-valued logic

- In the set-valued extension of the 3-valued logic:
  - in addition to the six sets corresponding to interval-valued extension of this logic,
  - we have two more sets.
- We have the empty set $\emptyset$ corresponding to situations in which no expert has any opinion.
- We have the set $\{0, 1\}$ corresponding to the polarized case when:
  - some experts strongly believe that the given statement is true,
  - while others as strongly believe that this statement is false – case typical in politics.

19. Set-valued extension of 3-valued logic naturally leads to color optical computing

- In color optical computing, we start with three basic colors read (R), green (G), and blue (B).
- Their position on the spectrum is described as $R < G < B$.
- We also consider combinations of some of these colors, i.e., all subsets of the set $\{R, G, B\}$.
- We can have three pure colors corresponding to three 1-element sets $\{R\}$, $\{G\}$, and $\{B\}$.
- We can have white – a combination of all three basic colors – corresponding to the set $\{R, G, B\}$.
- We can have black – where there are no colors at all – corresponding to the empty set.
- We can also have combinations of two of three colors.

20. Set-valued extension of 3-valued logic naturally leads to color optical computing (cont-d)

- These $2^3 = 8$ combinations are in natural 1-to-1 correspondence with eight subsets that form the set-valued extension of the 3-valued logic.
- This provides a natural explanation of the color optical interpretation of fuzzy logic.

21. Conclusions

- In the classical logic, every statement is either true or false.
- In a computer, "true" is usually represented by 1, and "false" by 0.
- In many practical situations, we are unsure whether the statement is true or false.
- To describe different degrees of confidence in a statement, Lotfi Zadeh proposed to use real numbers between 0 and 1.
- From the purely mathematical viewpoint, there are infinitely many real numbers between 0 and 1.
- However, we humans can only meaningfully distinguish between a small number of different degrees of confidence; thus:
  - to make the description of degrees of confidence more adequate,
  - it makes sense to restrict ourselves to finite (discrete) subsets of the interval $[0, 1]$.

22. Conclusions (cont-d)

- To make this description even more adequate:
  - it is desirable to also take into account that sometimes,
  - experts are unsure which of the possible degrees better describe their degree of confidence.
- To cover such situations, we need to consider:
  - subsets of the set of possible degrees,
  - i.e., set-valued extensions of discrete fuzzy logics.
- An important particular case is an interval-valued extension, when:
  - we only consider intervals,
  - i.e., the sets of all the degrees between two bounds.

23. Conclusions (cont-d)

- It turns out that:
  - these extension ideas naturally lead to several known effective techniques,
  - and thus, provide an explanation for these techniques' effectiveness.
- The set-theoretic extension of the 2-valued logic naturally leads to the known technique of Belnap's logic.
- This technique enables us to allow knowledge bases with inconsistencies.
- The set-theoretic extension of the 3-valued discrete fuzzy logic naturally leads to color optical computing.
- This is an empirically successful way of representing and processing fuzzy degrees by different colors.

24. Acknowledgments

This work was supported in part by:

- National Science Foundation grants 1623190, HRD-1834620, HRD-2034030, and EAR-2225395;
- AT&T Fellowship in Information Technology;
- program of the development of the Scientific-Educational Mathematical Center of Volga Federal District No. 075-02-2020-1478, and
- a grant from the Hungarian National Research, Development and Innovation Office (NRDI).