How to Share a Success, How to Share a Crisis, and How All This Is Related to Fuzzy

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1. How to share a success: a problem

- Often, a group of people has an opportunity to benefit all its members.
- For example, family members get an inheritance in which their late relative did not specify who gets what.
- In many such cases, there are many possible alternative decisions.
- In some of these possible alternatives, some of the participants benefit more, in others, other participants benefit more.
- Which of these alternatives should we choose?

2. Known solution: Nash's bargaining solution and its relation to fuzzy

- This problem is well studied in decision theory.
- The solution to this problem was provided by John Nash – who later won a Nobel prize for his research.
- He showed that:
  - under some reasonable requirements (that we will describe later),
  - the group should select an alternative for which the product $U_1 \cdot U_2 \cdot \ldots \cdot U_n$ of their utility gains U_i is the largest possible.
- This solution is known as *Nash's bargaining solution.*
- This solution has a natural interpretation in fuzzy logic.
- Indeed, it is reasonable to make sure that everyone is as happy as possible; the 1st person is happy *and* the 2nd person is happy, etc.

3. Nash's bargaining solution and its relation to fuzzy (cont-d)

- Thus, it makes sense to select an alternative for which:
  - our degree of confidence in the statement "the 1st person is happy and the 2nd person is happy, etc."
  - is the largest possible.
- It is natural to use utility gain U_i as the measure of happiness.
- To combine these degrees, we can use one of the most widely used "and"-operations: algebraic product.
- Then, our degree of confidence that an alternative makes everyone happy is equal to the product of utilities.
- This is exactly what Nash's bargaining solution is about.

4. But what if there is a crisis?

- Nash's bargaining solution is only applicable in situations of success, when everyone gains.
- But sometimes, we encounter the opposite situation – of a crisis, when everyone needs to sacrifice.
- For example, there is an overall budget cut, and some salaries need to be cut: what is the fair way to do it?
- In general, what is the fair way to share a crisis?

5. What we do in this talk

- In this talk, we analyze what is the fair way to share a crisis, i.e., a situation when we need to decrease utilities.
- Our starting point is utility-based Nash's bargaining solution.
- So, we start the talk by reminding the reader what is utility and how Nash's bargaining solution is justified.
- Then, we show that a similar approach – based on natural requirements – does not work for the case of a crisis.
- After that, we provide a different set of requirements and show that it enables us to come up with an (almost) unique fair solution.
- Finally, if we have time, we present the proofs.

6. What is utility

- One of the main objectives of decision theory is to help people make decisions in complex situations.
- In situations in which there is a very large number of alternatives, it is not possible for a person to process all this data by hand.
- So, we need to use computers.
- Information about different alternatives comes in different formats, with words, etc.
- Computers, however, are not very good in processing words, they are much better in processing numbers.
- This is what they were originally designed for.
- So, to effectively use computers, we need to describe all the available information in numerical terms.
- In particular, we need to describe people's preferences in numerical form.

7. What is utility (cont-d)

- For this description, the notion of utility was invented.
- This notion allows us:
  - to assign, to each alternative a, a number $u(a)$ – called its *utility*,
  - so that a is preferable to b if and only if $u(a)$ is larger than $u(b)$.
- To describe the utilities, we need to select two extreme alternatives, ideally not realistic:
  - a very bad alternative a_- which is worse than any actual alternatives, and
  - a very good alternative a_+ which is better than any actual alternative.
- Once these alternatives are selected, we can form, for each value p from the interval $[0, 1]$, a *lottery* $L(p)$ in which:
  - we get a_+ with probability p, and
  - we get a_- with the remaining probability $1 - p$.

8. What is utility (cont-d)

- Of course, the larger the probability p of getting a very good outcome, the better the lottery.
- If $p > p'$ then, for the user, the lottery $L(p)$ is better than the lottery $L(p')$.
- We will denote this preference by $L(p) \succ L(p')$.
- To find a numerical value $u(a)$ corresponding to an alternative a, we need to compare a with lotteries $L(p)$ corresponding to different values $p \in [0, 1]$.
- For each p:
  - either a is better ($a \succ L(p)$),
  - or the lottery is better ($L(p) \succ a$),
  - or the alternative has the same value to the user as the lottery; we will denote this by $a \sim L(p)$.

9. What is utility (cont-d)

- When p is small, the lottery $L(p)$ is close to the very bad alternative a_- and is, therefore, worse than a: $a \succ L(p)$.
- On the other hand, when p is close to 1, the lottery $L(p)$ is close to the very good alternative a_+ and is, therefore, better than a: $L(p) \succ a$.
- One can show that there exists a threshold value $u(a)$ that separates the values p for which $a \succ L(p)$ from the values p for which $L(p) \succ a$.
- This value is equal to

$$u(a) = \sup\{p : a \succ L(p)\} = \inf\{p : L(p) \succ a\}.$$

- This threshold value is called the utility of the alternative a.
- It can be proven that for any set of alternatives $a_1, \ldots, a_n$:
  - if we consider a lottery in which we get a_i with probability p_i,
  - then the utility of this lottery is equal to $p_1 \cdot u(a_1) + \ldots + p_n \cdot u(a_n)$.

10. Comments

- At first glance, this notion may appear to be not very practical:
  - there are infinitely many possible values p, s
  - so comparing the alternative a with all these lotteries may take forever.
- However, this is not an obstacle: we can find the utility value really fast if we use bisection.
- Namely, we start with the interval $[\underline{u}, \overline{u}] = [0, 1]$ that contains the actual (unknown) value $u(a)$.
- At each iteration, we decrease this interval – while making sure that the shrank interval still contains $u(a)$.
- Namely, we compute the midpoint $\widetilde{u}$ of the current interval, and compare the alternative a with the lottery $L(\widetilde{u})$.

11. Comments (cont-d)

- If a is better than $L(\widetilde{u})$, then $u(a)$ is located in the interval $[\widetilde{u}, \overline{u}]$.
- If a is worse than $L(\widetilde{u})$, then $u(a)$ is located in the interval $[\underline{u}, \widetilde{u}]$.
- On each iteration, we make one comparison, and the interval becomes twice narrower.
- In k iterations, we thus get the interval of width 2^{-k}.
- We stop when the width of this interval because smaller than a given accuracy ε.
- This way, the midpoint of the resulting interval approximates $u(a)$ with accuracy $\varepsilon/2$.
- So, in 6 iteration, we reach accuracy 1%, and in 9 iterations, we reach accuracy 0.1%.

12. Comments (cont-d)

- Please note that the notion of utility is simply a reflection of the person's preferences.
- It does not mean that this person only cares about him/herself.
- In general, a person's preferences depend not only on this person's gains or losses, they are affected by gains and losses of others.
- Accordingly, the person's utility of each alternative depends not only on this person's gains and losses.
- It also depends on gains and losses of others.

13. Utility is defined modulo a strictly increasing linear transformation

- The above definition of utility depends on the selection of the two extreme alternatives a_- and a_+.
- If we select a different pair (a'_-, a'_+), then, in general, we get different numerical values of the utility.
- It can be proven that the new utility values $u'(a)$ can be obtained from the original ones $u(a)$ by a strictly increasing linear transformation.
- In precise terms, there exist constants $c > 0$ and d for which, for every alternative a, we have $u'(a) = c \cdot u(a) + d$.

14. Nash's bargaining solution: natural requirements and the resulting criterion

- In the success situations, we start with some starting state, which is known as the *status quo* state.
- In this state, the participants' utilities form a tuple $s \stackrel{\text{def}}{=} (s_1, \ldots, s_n)$.
- We have the set S of different possible alternative in which everyone gains, i.e., in which for the resulting utilities $u = (u_1, \ldots, u_n)$ we have:

$$u_i > s_i \text{ for all } i.$$

- For each two possible alternatives u and u', it is also possible:
  - for each value $p \in [0, 1]$,
  - to have a lottery in which we get u with probability p and u' with the remaining probability $1 - p$.
- As we have mentioned, the utility of this lottery is equal to

$$p \cdot u + (1 - p) \cdot u'.$$

15. Nash's bargaining solution: natural requirements and the resulting criterion (cont-d)

- This is an example of a convex combination of the two vectors u and u'.
- Thus, the set S should contain, with every two vectors, its convex combination – i.e., S should be a convex set.
- We need to come up with a group-based preference relation $\succ_s$ between the tuples.
- Let us list natural requirements.
- First is what is called *Pareto optimality*: if $u_i > u'_i$ for all i (we will denote it by $u > u'$), then we should have $u \succ_s u'$.
- Second, the preference relation should only depend on the preferences, it should not depend on the choice of a_- and a_+ for each person.

16. Nash's bargaining solution: natural requirements and the resulting criterion (cont-d)

- In other words, for every two tuples $c = (c_1, \ldots, c_n)$ with $c_i > 0$ for all i and $d = (d_1, \ldots, d_n)$:
  - if we denote $T_{c,d}(u) \stackrel{\text{def}}{=} (c_1 \cdot u_1 + d_1, \ldots, c_n \cdot u_n + d_n)$,
  - then $u \succ_s u'$ should imply $T_{c,d}(u) \succ_{T_{c,d}(s)} T_{c,d}(u')$.
- Third, preferences should not depend on how we number the participants.
- If we swap i and j – we will denote this transformation by $\pi_{i,j}$ – then preference should not change, i.e., $u \succ_s u'$, then $\pi_{i,j}(u) \succ_{\pi_{i,j}(s)} \pi_{i,j}(u')$.

17. Nash's bargaining solution: natural requirements and the resulting criterion (cont-d)

- Finally, the solution should be fair: equal participants should get equal benefit.
- In precise terms: if both the status quo state s and the set S do not change under the swap, i.e., if $\pi_{i,j}(s) = s$ and $\pi_{i,j}(S) = S$, then:
  - for every vector $u \in S$
  - there should exist a vector u' which is either better or of the same quality as u (we will denote it by $u' \succeq_s u$) for which $\pi_{i,j}(u') = u'$.
- Let us describe these conditions in precise terms.

18. Definition and notations

- Let $n > 1$ be an integer.
- A binary relation $\succeq$ on a set A is called a *total pre-order* if it satisfied the following three conditions for all a, b, and c:
    - if $a \succeq b$ and $b \succeq c$, then $a \succeq c$ (*transitivity*),
    - $a \succeq a$ (*reflexivity*), and
    - $a \succeq b$ or $b \succeq a$ (*totality*).
- For each such relation:
    - if $a \succeq b$ and $b \not\succeq a$, we will denote it by $a \succ b$, and
    - if $a \succeq b$ and $b \succeq a$, we will denote it by $a \sim b$.

19. Definition

- Let $n > 1$ be an integer.
- We say that we have a *preference relation* if for every vector $s \in \mathbb{R}^n$, we have total pre-order $\preceq_s$ on the set of all tuples x for which $x > s$.
- We say that a preference relation is:
  - *Pareto-optimal* if $u > u$ implies $u \succ u'$;
  - *scale-invariant* if for every two tuples $c = (c_1, \ldots, c_n)$ with $c_i > 0$ for all i and $d = (d_1, \ldots, d_n)$, $u \succeq_s u'$ implies that:

    $$T_{c,d}(u) \succeq_{T_{c,d}(s)} T_{c,d}(u'), \text{ where } T_{c,d}(u) \stackrel{\text{def}}{=} (c_1 \cdot u_1 + d_1, \ldots, c_n \cdot u_n + d_n);$$
  - *anonymous* if for every i and j, $u \succeq_s u'$ implies $\pi_{i,j}(u) \succeq_{\pi_{i,j}(s)} \pi_{i,j}(u')$, where $\pi_{i,j}$ swaps elements u_i and u_j in a vector.

20. Definition (cont-d)

- We say that a preference relation is *fair* if for every vector s and for every convex set S all of whose elements x satisfy the condition $x > s$:
  - once $\pi_{i,j}(s) = s$ and $\pi_{i,j}(S) = S$,
  - then for every $u \in S$ there exists a vector u' which is either better or of the same quality as u and for which $\pi_{i,j}(u') = u'$.

21. Proposition 1

For every n, for every preference relation $\succeq_s$, the following two conditions are equivalent:

- *the preference relation is Pareto-optimal, scale-invariant, anonymous, and fair;*
- *the preference relation has Nash's form*

$$u \succeq_s u' \leftrightarrow \prod_{i=1}^{n} U_i \geq \prod_{i=1}^{n} U'_i, \text{ where } U_i \stackrel{\text{def}}{=} u_i - s_i \text{ and } U'_i \stackrel{\text{def}}{=} u'_i - s_i.$$

22. Comment

- In our proof, we will show that we do not actually need the fairness condition.
- In this case, it follows from the other three conditions.
- However, we keep this reasonable condition in the definition.
- The reason is that, as we show later, in the crisis case, it does not automatically follow from the other conditions.

23. Can a Similar Approach Find a Fair Way to Share a Crisis?

- Let us now consider the case of a crisis, when we can no longer maintain the status quo level.
- In this case, fairness means that everyone should contribute, i.e., that we only consider vectors x for which $u_i < s_i$ for all i.
- It seems reasonable to make similar requirements about preferences as in the case of success – Pareto-optimality, scale-invariance, etc.
- Unfortunately, in this case, it is not possible to satisfy all these four conditions.

24. Definition

- Let $n > 1$ be an integer.
- We say that we have a *crisis-related preference relation* if:
  - for every vector $s \in \mathbb{R}^n$,
  - we have a total pre-order relation $\preceq_s$ on the set of all tuples x_i for which $x_i < s_i$ for all i.

25. Proposition 2 and comments

- *No crisis-related preferences relation is Pareto-optimal, scale-invariant, anonymous, and fair.*
- As we can see from the proof:
  - if we do not impose the fairness condition,
  - then the smaller the product of losses $s_i - u_i$, the better.
- In this case, there is no best outcome.
- However, we can get as close to the best if:
  - we let at least one participant to keep almost everything, i.e., to have $u_i \approx s_i$.
  - while others will suffer.
- This is clearly not a fair solution.

26. So What Is a Fair Way to Share a Crisis?

- The above negative result shows that we cannot come up with a fair solution if all we know is the current state – the status quo state s.
- So, to come up with a fair solution, a natural idea is to also take into account the state s' at some previous moment of time.
- So, we need a mapping – or maybe several possible mappings – that will:
  - given two vectors s and s',
  - compute the reduced-gain vector u, with $u_i < s_i$.
- Similarly to the case of sharing a gain, it makes sense to require scale-invariance.
- Thus, we arrive at the following definition.

27. Definition

- By *crisis-related decision function*, we mean a continuous function $F(s, s')$ that transforms pair of tuples into a new tuple s'' for which

$$s'' \le s.$$

- We say that the decision function is:
  - *scale-invariant* if for every two tuples $c > 0$ and d, $s'' = F(s, s')$ implies $T_{c,d}(s'') = F(T_{c,d}(s), T_{c,d}(s'))$;
  - *anonymous* if for every i and j, $s'' = F(s, s')$ implies

$$\pi_{i,j}(s'') = F(\pi_{i,j}(s), \pi_{i,j}(s')).$$

28. Proposition 3

For each crisis-related decision function $F(s, s')$, the following two conditions are equivalent to each other:

- *the function $F(s, s')$ is scale-invariant and anonymous, and*
- *there exist values $\alpha_+ \geq 0$ and $\alpha_- \geq 0$ for which, for all s and s', the components of the tuple $s'' = F(s, s')$ have the following form:*
  - $s''_i = s_i - \alpha_+ \cdot (s_i - s'_i)$ *when* $s_i \leq s'_i$ *and*
  - $s''_i = s_i - \alpha_- \cdot (s'_i - s_i)$ *when* $s'_i \leq s_i$.

29. Discussion

- When for all i, we have $s'_i \le s_i$, then we only need to use α_+.
- The smaller α_+, the better.
- Thus, in this case, Proposition uniquely determines the optimal strategy: we need to select the smallest possible value α_+.
- Similarly, when for all i, we have $s'_i \le s_i$, then we only need to use the parameter α_-.
- The smaller α_-, the better.
- Thus, in this case, Proposition 3 uniquely determines the optimal strategy: we need to select the smallest possible value α_-.
- In the general case, for some i, we have $s'_i < s_i$ while for other indices j, we have $s_j < s'_j$.

30. Discussion (cont-d)

- Then, we have a whole family of possible solutions.
- Namely, a 1-D family corresponding to Pareto-optimal solutions, i.e., in this case, pairs (α_+, α_-).

31. Proof of Proposition 1, Part 1

- It is easy to see that the Nash's preference relation satisfies the first three conditions.
- To get the fourth condition, it is sufficient to take the vector

$$u' = 0.5 \cdot u + 0.5 \cdot \pi_{i,j}(u).$$

- This is a vector in which we replace both values u_i and u_j with their arithmetic average.
- Indeed, in this case, we keep all the other terms in the product intact and replace the product $U_i \cdot U_j$ with the value $U_i' \cdot U_j'$, where

$$U_i' = U_j' = \frac{U_i + U_j}{2}.$$

- One can show that $\left(\dfrac{U_i + U_j}{2}\right)^2 \geq U_i \cdot U_j$.

32. Proof of Proposition 1, Part 1 (cont-d)

- Indeed, the difference between the left-hand side and the right-hand side is equal to $\left(\dfrac{U_i - U_j}{2}\right)^2 \geq 0$.
- So:
  - to complete the proof of Proposition 1,
  - it is sufficient to prove that every preference relation that satisfies the first three conditions has the Nash's form.

33. Proof of Proposition 1, Part 2

- Let us show that because of scale-invariance, we can reduce the family of total pre-orders to a single total pre-order.
- Indeed, for $c_i = 1$ for all i and $d = -s$, scale-invariance implies that $u \succeq_s u'$ if and only if $U \succeq_0 U'$, where $U \stackrel{\text{def}}{=} u - s$ and $U' \stackrel{\text{def}}{=} u' - s$.

34. Proof of Proposition 1, Part 3

- Let us now prove that for every U, we have $\pi_{i,j}(U) \sim_0 U$.
- Indeed, since the relation $\sim_0$ is total, we have either $\pi_{i,j}(U) \sim_0 U$, or $\pi_{i,j}(U) \succ_0 U$, or $U \succ_0 \pi_{i,j}(U)$.
- In the second case, anonymity would lead to $\pi_{i,j}(\pi_{i,j}(U)) \succ_0 \pi_{i,j}(U)$, i.e., to $U \succ_0 \pi_{i,j}(U)$, which contradicts to $\pi_{i,j}(U) \succ_0 U$.
- In the third case, anonymity would lead to $\pi_{i,j}(U) \succ_0 \pi_{i,j}(\pi_{i,j}(U))$, i.e., to $\pi_{i,j}(U) \succ_0 U$, which contradicts to $U \succ_0 \pi_{i,j}(U)$.
- Thus, the only remaining option is $\pi_{i,j}(U) \sim_0 U$.

35. Proof of Proposition 1, Part 4

- Let us now prove that:
  - for every vector U and for all i and j,
  - U is equivalent to a vector U' in which both components U_i and U_j are replaced by their geometric mean $\sqrt{U_i \cdot U_j}$.
- Indeed, due to Part 3 of this proof, we have

$$(\ldots, \sqrt{U_i}, \ldots, \sqrt{U_j}, \ldots) \sim_0 (\ldots, \sqrt{U_j}, \ldots, \sqrt{U_i}, \ldots).$$

- Due to scale-invariance for $d = 0$, $c_i = \sqrt{U_i}$, $c_j = \sqrt{U_j}$, and $c_k = 1$ for all other k, we indeed conclude that

$$(\ldots, U_i, \ldots, U_j, \ldots) \sim_0 (\ldots, \sqrt{U_i \cdot U_j}, \ldots, \sqrt{U_i \cdot U_j}, \ldots).$$

36. Proof of Proposition 1, Part 5

- Let us now prove that every vector U is equivalent to the vector consisting of n geometric means $\overline{U} \stackrel{\text{def}}{=} \sqrt[n]{U_1 \cdot \ldots \cdot U_n}$.
- Indeed, we will use Part 4 of this proof to prove:
  - by induction over $i = 0, 1, \ldots, n$, that
  - for each i, the vector U is equivalent to some vector of the type $(\overline{U}, \ldots, \overline{U}, U'_{i+1}, \ldots)$ in which the first i terms are equal to $\overline{U}$.
- The base case is easy: for $i = 0$, as the desired vector, we can take the same vector U.
- The induction step is as follows.
- Let us assume that we have such a representation for i:

$$U \sim_0 (\overline{U}, \ldots, \overline{U}, U'_{i+1}, U'_{i+2}, \ldots).$$

37. Proof of Proposition 1, Part 5 (cont-d)

- Then, due to Part 4 of the proof, we have

$$(\overline{U}, \ldots, \overline{U}, U'_{i+1}, U'_{i+2}, U'_{i+3} \ldots) \sim_0 (\overline{U}, \ldots, \overline{U}, \overline{U}, U''_{i+2}, U'_{i+3} \ldots).$$

- This holds as long as $U'_{i+1} \cdot U'_{i+2} = \overline{U} \cdot U''_{i+2}$.
- So this equivalence holds for $U''_{i+1} = \dfrac{U'_{i+1} \cdot U'_{i+2}}{\overline{U}}$.
- The induction is proven.
- So, for $i = n$, we get the desired result.

38. Proof of Proposition 1, Part 6

- So, due to Part 5, every vector U is equivalent to a vector $(\overline{U},\ldots,\overline{U})$, where $\overline{U}$ is the n-th root of the product of the values U_i.
- Thus, every two vectors with the same product are equivalent to each other.
- Due to Pareto optimality, if the product is larger, the vector is better.
- So indeed, the preference relation that satisfies the first three condition has the Nash's form.
- The proposition is proven.

39. Proof of Proposition 2

- Similarly to the proof of Proposition 1, we can prove that:
  - under the first three conditions,
  - every tuple U is equivalent to a tuple $(\overline{U}, \ldots, \overline{U})$, where $\overline{U} = \sqrt[n]{U_1 \cdot \ldots \cdot U_n}$, where this time $U_i \stackrel{\text{def}}{=} s_i - u_i$.
- Due to Pareto optimality, the smaller $\overline{U}$, the better.
- So the only preference relation that satisfies the first three conditions is the relation $u \succeq_s u' \leftrightarrow \overline{U} \geq \overline{U}'$.
- To complete the proof, we will show that this preference relation does not satisfy the fairness condition.
- Indeed, let us take $s = 0$ and

$$S = \{(-x, -(3 - x), -1, \ldots, -1) : 0 < x < 3\}.$$

- Both the status quo state and the set S do not change if we swap participants 1 and 2.

40. Proof of Proposition 2 (cont-d)

- For the vector $u = (-2, -1, -1, \ldots, -1)$, we have $\overline{U} = \sqrt[n]{2}$.
- However, the only outcome which is invariant with respect to the 1-2 swap is $u' = (-1.5, -1.5, -1, \ldots, -1)$.
- For this outcome, $\overline{U}' = \sqrt[n]{1.5 \cdot 1.5} = \sqrt[n]{2.25} > \overline{U}$.
- Thus, here $u \succ_s u'$ – and hence, the preference relation violates the fairness condition.
- The proposition is proven.

41. Proof of Proposition 3, Part 1

- It is easy to show that for each $\alpha_+ \geq 0$ and $\alpha_- \geq 0$, the corresponding function is scale-invariant and anonymous.
- So, to complete the proof, we need to show that, vice versa, every scale-invariant anonymous function has the desired form.

42. Proof of Proposition 3, Part 2

- Due to scale-invariance for $c_i = 1$ and $d_i = -s_i$, $s'' = F(s, s')$ implies that $s'' - s = F(0, s' - s)$.
- So, $s'' = F(s, s') = G(s' - s) + s$, where we denoted $G(U) \stackrel{\text{def}}{=} F(0, U)$.
- Thus, to describe all possible scale-invariant anonymous functions $F(s, s')$, it is sufficient to describe functions $G(U) = F(0, U)$.
- Since $T_{c,0}(0) = 0$, for this new function, scale-invariance means that for every $c > 0$, if $U' = G(U)$, then $T_c(U') = G(T_c(U))$.
- In other words, we have $T_c(G(U)) = G(T_c(U))$.

43. Proof of Proposition 3, Part 3

- When $U_i = 0$ for some i, then for $c_i = 2$ and $c_j = 1$ for all other j, we have $T_c(U) = U$.
- Thus, scale-invariance implies that $T_c(G(U)) = G(U)$.
- For the i-th component of the vector $U' = G(U)$, this means $U'_i = 2U'_i$, thus $U'_i = 0$.

44. Proof of Proposition 3, Part 4

- For the values i for which $U_i \neq 0$, we can use $c_i = 1/|U_i|$.
- Then, the vector $e \stackrel{\text{def}}{=} T_c(U)$ contains only components that are equal to 1, -1, and 0.
- Let n_- be the number of values equal to -1, n_+ equal to the number of values equal to 1, and n_0 be the number of values equal to 0.
- For every triple $N = (n_+, n_-, n_0)$ different vectors e corresponding to this case can obtained from each other by permutation.
- Thus, due to anonymity:
  - all n_+ participants i with $e_i = 1$ get the same value t_i which we will denote by $\alpha_+(N)$, and
  - all n_+ participants i with $e_i = -1$ get the same value t_i which we will denote by $\alpha_-(N)$, and
- By using scale-invariant, we can conclude that the values U'_i have the desired form.

45. Proof of Proposition 3, Part 4 (cont-d)

- The only difference is that now, the values α_+ and α_-, in general, depends on the vector N.
- To complete the proof, we need to prove that the values α_+ and α_- are the same for al triples N.

46. Proof of Proposition 3, Part 5

- Let us pick the value i for which $e_i = 1$.
- Let us consider a family of vectors that are obtained:
  - by multiplying all the values U_j (except for U_i) by some value ε,
  - while U_i remains intact.
- For all $\varepsilon > 0$, we have the same vector N.
- So we have the same formulas for U'_i – with the coefficients α_+ corresponding to this vector N.
- In the limit $\varepsilon \to 0$, s''_i tends to a vector in which there is only one non-zero component.
- For this limit vector, the corresponding vector N' has the form $(1, 0, n - 1)$.
- The function $F(s, s')$ is continuous.
- So, the value s''_i corresponding to the limit vector N' should be equal to the limit of the values corresponding to N.

47. Proof of Proposition 3, Part 5 (cont-d)

- For all $\varepsilon > 0$, the value s''_i is the same – corresponding to $\alpha_+(N)$.
- So, in the limit, this value should remain the same as well.
- However:
  - for $\varepsilon > 0$, we have $\alpha_+(N)$ corresponding to the original vector N,
  - while in the limit, we have the value $\alpha_+(N')$.
- Thus, for each vector N, the value $\alpha_+(N)$ is the same as in the case when only coefficient is different from 0.
- So $\alpha_+(N)$ does not depend on N.
- Similarly, the value $\alpha_-(N)$ does not depend on N.
- The proposition is proven.

48. Conclusions

- How can we share a success?
- How can we fairly divide the gains between people who contributed to this gain?
- Decision theory provides an answer to this question.
- Namely, as shown by the Nobelist John Nash, reasonable conditions uniquely determine the fair division.
- This division is known as Nash's bargaining solution.
- It has a natural interpretation in fuzzy terms, as a solution for which:
  - the desired statement – that all participants are happy
  - is satisfied to the largest degree.

49. Conclusions (cont-d)

- At first glance, it may seem that a similar approach can be applied to crisis situations:
  - when there is a need for sacrifices – e.g., for salary cuts – and
  - we are looking for the most fair way to share the crisis.
- Somewhat surprisingly, we show that in the case of a crisis, no solution satisfies all Nash's requirements.
- We then describe a weaker set of requirements – that (almost) uniquely determine how to share a crisis.

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