Computational Aspects of Physical Models Based on Berwald-Moore-based Finsler Geometry: General Computational Complexity and Specifics of Relativistic Celestial Mechanics Testing

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1. Introduction: Main Ideas Behind Special Relativity

- Special relativity started with two principles.
- *Relativity principle:* all inertial motions are physically equivalent.
- *Additional idea:* all physical velocities are limited by the speed of light c: $|\vec{v}| \le c$.
- *Special relativity:* an event $e = (t, x)$ can causally influence an event $e' = (t', x')$ ($e \preceq e'$) if it is possible,
  - starting at location x at moment t,
  - reach location x' at moment t',
  - while traveling at speed $|\vec{v}| \le c$.
- *Resulting formula:* $\dfrac{d(x, x')}{t' - t} \le c$

2. Symmetry of Relativity Theory

- *Reminder:* $e = (t, x) \preceq e' = (t', x') \Leftrightarrow \dfrac{d(x, x')}{t' - t} \le c.$
- *Resulting formula:* $t' > t$ and $c^2 \cdot (t' - t)^2 - d^2(x, x') \ge 0.$
- *Kinematic causality relation:* generated by moving bodies (with non-zero rest mass), for which $|\vec{v}| < c$.
- *Formula:* $t' > t$ and $c^2 \cdot (t' - t)^2 - d^2(x, x') > 0.$
- *Symmetry:* the future cone is *homogeneous*:
  - for every three events e, e', e'' for which $e \prec e'$ and $e \prec e''$,
  - there exists a causality-preserving transformation that transforms (e, e') into (e, e'').

3. From Special Relativity to More Physical Curved Space-Times

- *Causality (reminder):*

$$e = (t, x) \preceq e' = (t', x') \Leftrightarrow \frac{d(x, x')}{t' - t} \leq c.$$

- Einstein and Minkowski proposed the pseudo-Euclidean Minkowski *metric*

$$s^2((t, x), (t', x')) = c^2 \cdot (t' - t)^2 - d^2(x, x').$$

- *Fact:* this metric forms the basis of the current Riemannian-geometry-based physical theories of space-time.
- *Problems:* from the physical viewpoint, there are still many problems with current space-time models.
- *Result:* search for more general geometrical models.
- *Reasonable idea:* search for a basic space-time model which is different from Minkowski space.

4. Towards a General Symmetric Space-Time Model

- *Objective:* find a general basic space-time model.
- *Main physical requirement:* basic symmetries:
  - shift-invariance,
  - scale-invariance,
  - homogeneous (= satisfies relativity principle).
- *Mathematical description:* a "flat" space-time (R^n) in which the causality relation is:
  - shift-invariant: $e \prec e' \Leftrightarrow e + e'' \prec e' + e''$;
  - scale-invariant: $e \prec e' \Leftrightarrow \lambda \cdot e \prec \lambda \cdot e'$; and
  - homogeneous:
    * for every three events e, e', e'' for which $e \prec e'$ and $e \prec e''$,
    * there exists a causality-preserving transformation that transforms (e, e') into (e, e'').

5. Classification of Symmetric Space-Time Models

- *Reminder:* we are looking for orderings $\prec$ of R^n which are:
  - shift- and scale-invariant and
  - homogeneous on the future cone $\{e' : e \prec e'\}$.
- *Classification theorem* (A.D. Alexandrov): each such space-time is
  - either a Minkowski space,
  - or a Cartesian product $X_1 \times \ldots \times X_n$ of Minkowski spaces of smaller dimension:

$$(x_1, \ldots, x_n) \prec (x'_1, \ldots, x'_n) \Leftrightarrow (x_1 \prec x'_1) \,\&\, \ldots \,\&\, (x_n \prec x'_n).$$

- *4-D example:* the product of 4 subspaces R is R^4 with the ordering $x \prec x'$ iff $x_i < x'_i$ for all i.

6. A New Model: Symmetries and Metrics

- *Ordering (reminder):* $x \prec x'$ iff $x_i < x'_i$ for all i.
- *General symmetries:* $x_i \to f_i(x_i)$.
- *Symmetries consistent with shift and scaling:*

$$x_i \to a_i \cdot x_i + b_i.$$

- *From ordering to a metric* $\tau(x, x')$ – requirements:
  - shift-invariant: $\tau(x, x') = \tau(x + x'', x' + x'')$;
  - scale-invariant: $\tau(\lambda \cdot x, \lambda \cdot x') = \lambda \cdot \tau(x, x')$;
  - homogeneous: under

$$T(x_1, \ldots, x_n) \stackrel{\text{def}}{=} (a_1 \cdot x_1 + b_1, \ldots, a_1 \cdot x_1 + b_1),$$

  we have $\tau(T(x), T(x')) = c(T) \cdot \tau(x, x')$.

- *Conclusion:* $\tau(x, x') = \left(\prod\limits_{i=1}^{4} (x'_i - x_i) \right)^{1/4}.$
- *Comment:* this is Berwald-Moore metric.

7. From the Basic Space-Time to General Space-Time Models

- *Traditional basis:* Minkowski metric.
- *Traditional extension:* spaces which are locally isomorphic to the Minkowski metric – pseudo-Riemannian spaces.
- *New basis:* Berwald-Moore metric.
- *Natural idea:* consider physical models of space-time based on this metric.
- *To be more precise:* models based on the Finsler spaces which are locally isomorphic to this metric.

8. Relation of Berwald-Moore Coordinates to Usual Physical Coordinates

- *Berwald-Moore coordinates:* a_i for which

$$\tau(a, a') = \left(\prod_{i=0}^{3}(a'_i - a_i)\right)^{1/4}.$$

- *Usual physical coordinates:* $x_0 = c \cdot t$ and x_i.
- *Relation – example:*

$$a_0 = x_0 + \frac{1}{\sqrt{3}} \cdot (x_1 + x_2 + x_3), \quad a_1 = x_0 + \frac{1}{\sqrt{3}} \cdot (x_1 - x_2 - x_3),$$

$$a_2 = x_0 + \frac{1}{\sqrt{3}} \cdot (-x_1 + x_2 - x_3), \quad a_3 = x_0 + \frac{1}{\sqrt{3}} \cdot (-x_1 - x_2 + x_3).$$

9. Computational Complexity of Prediction Problems

- *First problem:*
  - how the change in geometry
  - affects the computational complexity of the corresponding predictions.
- *Euclidean space* (result):
  - if we want to know all the distances with a given accuracy $\varepsilon > 0$,
  - then it is sufficient to find all the coordinates x_i of all the events with a similar accuracy $O(\varepsilon)$.

10. Euclidean Space: Proof

- *Result* (reminder):
  - to know all the distances with a given accuracy $\varepsilon > 0$,
  - it is sufficient to find all the coordinates x_i of all the events with a similar accuracy $O(\varepsilon)$.
- *Proof:*
  - triangle inequality implies that
$$|d(x,y) - d(x',y')| \le d(x,x') + d(y,y');$$
  - for Euclidean metric, we have
$$d(x,x') \le |x_1 - x'_1| + \ldots + |x_n - x'_n|;$$
  - hence, if $|x_i - x'_i| \le \varepsilon$ and $|y_i - y'_i| \le \varepsilon$, then $d(x,x') \le n \cdot \varepsilon$, $d(y,y') \le n \cdot \varepsilon$, and
$$|d(x,y) - d(x',y')| \le 2n \cdot \varepsilon.$$

11. From Euclidean to Minkowski Space

- *Euclidean space* (reminder):
  - if we want to know all the distances with a given accuracy $\varepsilon > 0$,
  - then it is sufficient to find all the coordinates x_i of all the events with a similar accuracy $O(\varepsilon)$.
- *Minkowski space:*
  - if we want to know all the distances with a given accuracy $\varepsilon > 0$,
  - we need to find all the coordinates x_i of all the events with a higher accuracy $O(\varepsilon^2)$.

12. Minkowski Space: Proof That Accuracy $O(\varepsilon^2)$ Is Sufficient

- *Minkowski space* (reminder):
  - if we want to know all the distances with a given accuracy $\varepsilon > 0$,
  - we need to find all the coordinates x_i of all the events with an accuracy $O(\varepsilon^2)$.
- *Proof:*
  - if we know coordinates x_i and x'_i with accuracy $O(\varepsilon^2)$,
  - then we can compute

    $$\tau^2(x, x') = (x_0 - x'_0)^2 - (x_1 - x'_1)^2 - \ldots - (x_n - x'_n)^2$$

    with accuracy $O(\varepsilon^2)$,
  - so, we can compute τ with accuracy $O(\varepsilon)$.

13. Minkowski Space: Proof That Accuracy $O(\varepsilon^2)$ Is Necessary

- *Minkowski space* (reminder):
  - if we want to know all the distances with a given accuracy $\varepsilon > 0$,
  - we need to find all the coordinates x_i of all the events with an accuracy $O(\varepsilon^2)$.
- *Proof:*
  - for $x' = (1+\delta, 1, 0, 0)$ and $x = (0, 0, 0, 0)$,
  - we have

$$\tau^2(x, x') = (1+\delta)^2 - 1 = 2\delta + o(\delta);$$

  - so $\tau(x, x') = \sqrt{2\delta} + o(\delta)$;
  - thus, to get τ with accuracy ε, we need $\delta \propto \varepsilon^2$.

14. From Minkowski to Berwald-Moore Space

- *Minkowski space:*
  - if we want to know all the distances with a given accuracy $\varepsilon > 0$,
  - we need to find all the coordinates x_i of all the events with an accuracy $O(\varepsilon^2)$.
- *Berwald-Moore space:*
  - if we want to know all the distances with a given accuracy $\varepsilon > 0$,
  - we need to find all the coordinates x_i of all the events with an even higher accuracy $O(\varepsilon^4)$.

15. Berwald-Moore Space: Proof That Accuracy $O(\varepsilon^4)$ Is Sufficient

- *Berwald-Moore space* (reminder):
  - if we want to know all the distances with a given accuracy $\varepsilon > 0$,
  - we need to find all the coordinates x_i of all the events with an accuracy $O(\varepsilon^4)$.
- *Proof:*
  - if we know coordinates x_i and x'_i with accuracy $O(\varepsilon^4)$,
  - then we can compute

    $$\tau^4(x, x') = (x'_0 - x_0) \cdot (x'_1 - x_1) \cdot (x'_2 - x_2) \cdot (x'_3 - x_3)$$

    with accuracy $O(\varepsilon^4)$,
  - so, we can compute τ with accuracy $O(\varepsilon)$.

16. Berwald-Moore Space: Proof That Accuracy $O(\varepsilon^4)$ Is Necessary

- *Berwald-Moore space* (reminder):
  - if we want to know all the distances with a given accuracy $\varepsilon > 0$,
  - we need to find all the coordinates x_i of all the events with an accuracy $O(\varepsilon^4)$.
- *Proof:*
  - for $x' = (\delta, 1, 1, 1)$ and $x = (0, 0, 0, 0)$,
  - we have

$$\tau^4(x, x') = (x'_0 - x_0) \cdot (x'_1 - x_1) \cdot (x'_2 - x_2) \cdot (x'_3 - x_3) =$$

$$(\delta - 0) \cdot (1 - 0) \cdot (1 - 0) \cdot (1 - 0) = \delta \cdot 1 \cdot 1 \cdot 1 = \delta;$$

  - so $\tau(x, x') = \delta^{1/4}$;
  - thus, to get τ with accuracy ε, we need $\delta \propto \varepsilon^4$.

17. Discussion

- *Reminder:* in the new model,
  - to predict all the distances with a given accuracy $\varepsilon > 0$,
  - we need to find all the coordinates x_i of all the events with a higher accuracy $O(\varepsilon^4)$ ($\ll O(\varepsilon^2)$).
- *Possible impression:* this result is negative.
- *Why:* it is an indication that for the new physical model, predictions are more computationally difficult.
- However, this same result can be interpreted positively.
- *Why:*
  - this result means that even small deviations of the events can lead to large differences in the metric;
  - therefore, in the new theory, it is easier to experimentally detect small effects.

18. Towards Analyzing Celestial Mechanics Effects

- *Optimistic viewpoint:* in the new theory, it is easier to experimentally detect small effects.
- From this viewpoint, we started analyzing possible celestial mechanical effects of the new geometry.
- At first glance, the metric is drastically different from the Minkowski one.
- Hence, for particles, we get a drastically different Lagrange function

$$\frac{ds}{dt} = \left[\left(1 + \frac{v_1 + v_2 + v_3}{\sqrt{3}}\right) \cdot \left(1 + \frac{v_1 - v_2 - v_3}{\sqrt{3}}\right) \cdot \ldots\right]^{1/4}.$$

- However, for planets and satellites, the new Lagrangian $\equiv$ the standard one $L_0 \stackrel{\text{def}}{=} 1 + \frac{1}{2}\vec{v}^2$ in quadratic terms.
- The only difference is in 4-th order terms.
- These terms can be analyzed similar to PPN.

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