To Properly Reflect Physicists' Reasoning about Randomness, We Also Need a Maxitive (Possibility) Measure

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1. Physicists assume that initial conditions and values of parameters are not abnormal

- To a mathematician, the main contents of a physical theory is its equations.
- Not all solutions of the equations have physical sense.
- *Ex. 1:* Brownian motion comes in one direction;
- *Ex. 2:* implosion glues shattered pieces into a statue;
- *Ex. 3:* fair coin falls heads 100 times in a row.
- *Mathematics:* it is possible.
- *Physics* (and common sense): it is not possible.
- *Our objective:* supplement probabilities with a new formalism that more accurately captures the physicists' reasoning.

2. A seemingly natural formalizations of this idea

- *Physicists:* only "not abnormal" situations are possible.
- *Natural formalization: idea.* If a probability $p(E)$ of an event E is small enough, then this event cannot happen.
- *Natural formalization: details.* There exists the "smallest possible probability" p_0 such that:
  - if the computed probability p of some event is larger than p_0, then this event can occur, while
  - if the computed probability p is $\le p_0$, the event cannot occur.
- *Example:* a fair coin falls heads 100 times with prob. 2^{-100}; it is impossible if $p_0 \ge 2^{-100}$.

3. The above formalization of the notion of "typical" is not always adequate

- *Problem:* every sequence of heads and tails has exactly the same probability.
- *Corollary:* if we choose $p_0 \geq 2^{-100}$, we will thus exclude all possible sequences of 100 heads and tails as physically impossible.
- However, anyone can toss a coin 100 times, and this proves that some such sequences are physically possible.
- *Similar situation:* Kyburg's lottery paradox:
  - in a big (e.g., state-wide) lottery, the probability of winning the Grand Prize is so small that a reasonable person should not expect it;
  - however, some people do win big prizes.

4. Relation to non-monotonic reasoning

- Traditional logic is *monotonic:* once a statement is derived it remains true.
- Expert reasoning is *non-monotonic:*
  - birds normally fly,
  - so, if we know only that Sam is a bird, we conclude that Sam flies;
  - however, if we learn the new knowledge that Sam is a penguin, we conclude that Sam doesn't fly.
- Non-monotonic reasoning helps resolve the lottery paradox (Poole et al.)
- *Our approach:* in fact, what we propose can be viewed as a specific non-monotonic formalism for describing rare events.

5. Events with 0 probabilities are possible: another explanation for the lottery paradox

- *Idea:* common sense intuition is false, events with small (even 0) probability are possible.
- This idea is promoted by known specialists in foundations of probability: K. Popper, B. De Finetti, G. Coletti, A. Gilio, R. Scozzafava, W. Spohn, etc.
- *Out attitude:* our objective is to formalize intuition, not to reject it.
- *Interesting:* both this approach and our approach lead to the same formalism (of maxitive measures).
- *Conclusion:* Maybe there is a deep relation and similarity between the two approaches.

6. Kolmogorov's idea: use complexity

- *Problem with the above naive approach:* we use the same threshold p_0 for all events.
- *Kolmogorov's idea:* the probability threshold $t(E)$ below which an event E is dismissed as impossible must depend on the event's complexity.
- The event E_1 in which we have 100 heads is easy to describe and generate; so $t(E_1)$ is higher.
- If $t(E_1) > 2^{-100}$ then, within this Kolmogorov's approach, we conclude that the event E_1 is impossible.
- On the other hand, the event E_2 corresponding to the actual sequence of heads and tails is much more complicated; so, $t(E_2)$ is lower.
- If $t(E_2) < 2^{-100}$, we conclude that the event E_2 is possible.

7. Towards Formalization

- *Original idea:* an event E is possible if and only its probability $p(E)$ exceeds a certain threshold p_0.
- *New idea:*
  - each event E has a "complexity" $c(E)$;
  - an event E is possible if and only if $p(E) > p_0 \cdot c(E)$.
- *Equivalent formulation:* E is possible of and only if $m(E) > p_0$, where $m(E) \stackrel{\text{def}}{=} p(E)/c(E)$ is a "ratio" measure.
- *Standard probability setting:*
  - Let X be the set of all possible outcomes.
  - An *event* is a subset E of the set X.
  - p is a probability measure on a σ-algebra $\mathcal{A}$ of sets from X.

8. Main result

- Let $T \subseteq X$ be the set of all outcomes that are actually possible.
- An event E is possible $\leftrightarrow$ there is a possible outcome that belongs to the set E, i.e., $\leftrightarrow E \cap T \neq \emptyset$.
- *Definition.* A *ratio measure* is a mapping from $\mathcal{A}$ to $[0, \infty]$ s.t. $\forall p_0 > 0\, \exists T(p_0)$ for which
$$\forall E \in \mathcal{A}\, (m(E) > p_0 \leftrightarrow E \cap T(p_0) \neq \emptyset).$$
- *Reminder:* m is a *maxitive* (*possibility*) measure if for every family of sets E_α
$$m\left(\bigcup_\alpha E_\alpha\right) = \sup_\alpha m(E_\alpha).$$
- *Theorem.* A function $m(E)$ is a ratio measure if and only if it is a maxitive (possibility) measure.

9. Discussion

- Our definition is slightly more general than usual:
  - *possibility* measures only use $m(E) \in [0,1]$;
  - *maxitive* measures only require finite families E_α.
- Since $m(E) = p(E)/c(E)$ is a possibility measure, we thus have $c(E) = m(E)/p(E)$. In other words,

$$\text{complexity} = \frac{\text{possibility}}{\text{probability}}.$$

- This result is in perfect accordance with a recent paper by D. Dubois, H. Fargier, and H. Prade.
- In that paper, the authors prove that the only uncertainty theory coherent with the notion of accepted belief is *possibility* theory.
- Moreover, even our proof is similar to the proofs from this recent paper.

10. Auxiliary result

- *Fact:* Our definition of complexity depends on the choice of the probability measure.
- *Question:* is it possible to have a complexity measure that will serve all possible probability measures $p(E)$?
- *Our answer, in brief:* "no", even if, instead of all possible thresholds p_0, we just consider a single one.
- Let $X = [0,1]$, and let $\mathcal{A} \subseteq 2^X$ be a σ-algebra of all Lebesgue-measurable sets.
- *Definition.* By a *universal complexity measure* c we mean a mapping $\mathcal{A} \to [0,1]$ for which $\forall a < b : 0 < c([a,b]) < 1$ and $\forall p, \exists T[p]$ s.t.

$$\forall E \in \mathcal{A}\,(p(E) > 0 \to (p(E) > c(E) \leftrightarrow E \cap T[p] \neq \emptyset)).$$

- *Theorem.* A universal complexity measure is impossible.

11. Conclusion

- According to the *traditional probability theory*, events with a positive but very small probability can occur (although very rarely).
- For example, from the *purely mathematical viewpoint*, it is possible that the thermal motion of all the molecules in a coffee cup goes in the same direction, so this cup will start lifting up.
- In contrast, *physicists believe* that events with extremely small probability cannot occur.
- In this paper, we show that to get a consistent formalization of this belief, we need,
  - in addition to the original *probability* measure,
  - to also consider a maxitive (*possibility*) measure.

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13. Proof: Part I

- Let us first prove that every ratio measure $m(E)$ is maxitive, i.e., $m(E) = \sup\limits_\alpha m(E_\alpha)$.
- By definition,

$$\forall p_0\, \exists T(p_0)\, \forall S(m(S) > p_0 \leftrightarrow S \cap T(p_0) > 0).$$

- *Why* $m(E) \not< \sup\limits_\alpha m(E_\alpha)$.
  - If $m(E) < \sup\limits_\alpha m(E_\alpha)$, let us select p_0 s.t.

$$m(E) < p_0 < \sup_\alpha m(E_\alpha).$$

  - Since $m(E) < p_0$, we conclude that $E \cap T(p_0) = \emptyset$.
  - On other hand, since $p_0 < \sup\limits_\alpha m(E_\alpha)$,

$$\exists \alpha_0\, (p_0 < m(E_{\alpha_0})).$$

  - For this α_0, there exists x from $E_{\alpha_0} \cap T(p_0)$.
  - However, since $E = \cup E_\alpha$, we have $x \in E$, so $x \in E \cap T(p_0)$ – a contradiction with $E \cap T(p_0) = \emptyset$.

14. Proof: Part I (cont-d)

- *Why* $m(E) \not> \sup\limits_\alpha m(E_\alpha)$.
  - If $m(E) > \sup\limits_\alpha m(E_\alpha)$, let us select p_0 s.t.

$$m(E) > p_0 > \sup_\alpha m(E_\alpha).$$

  - Since $m(E) > p_0$, there exists x in $E \cap T(p_0)$.
  - Since E is the union, $x \in E_{\alpha_0}$ for some α_0.
  - So, $E_{\alpha_0} \cap T(p_0) \neq \emptyset$.
  - Hence, $m(E_{\alpha_0}) > p_0$.
  - Therefore, $\sup\limits_\alpha m(E_\alpha) \geq m(E_{\alpha_0}) > p_0$ – a contradiction.
- *Conclusion:* every ratio measure is maxitive.

15. Proof: Part II

- Let us now prove that every maxitive measure is a ratio measure.
- We will prove it for

$$T(p_0) = -\cup\{S \in \mathcal{A} : m(S) \le p_0\}.$$

- We must prove that for every $E \in \mathcal{A}$,

$$E \cap T(p_0) \neq \emptyset \leftrightarrow m(E) > p_0.$$

- We actually prove an equivalent statement:

$$E \cap T(p_0) = \emptyset \leftrightarrow m(E) \le p_0.$$

- If $m(E) \le p_0$, then E is completely contained in the union

$$\cup\{S \in \mathcal{A} : m(S) \le p_0\}.$$

- Thus, E cannot have common points with the complement $T(p_0)$ to this union.

16. Proof: Part II (cont-d)

- Vice versa, if $E \cap T(p_0) = \emptyset$, then
$$E \subseteq \cup\{S \in \mathcal{A} : m(S) \leq p_0\}.$$
- Thus,
$$E = \cup\{S \cap E : S \in \mathcal{A} \,\&\, m(S) \leq p_0\}.$$
- For maxitive measures, from $S = (S \cap E) \cup (S - E)$, we conclude that $m(S) = \max(m(S \cap E), m(S - E)) \geq m(S \cap E)$.
- Hence, $m(S \cap E) \leq m(S) \leq p_0$.
- Thus, $m(E)$ is the supremum of a set of numbers each of which is $\leq p_0$.
- We can therefore conclude that $m(E) \leq p_0$.
- The theorem is proven.