Relation Between Polling and Likert-Scale Approaches to Eliciting Membership Degrees Clarified by Quantum Computing

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1. How Can We Elicit Membership Degrees?

- One of these methods is *polling:* we ask several experts whether, e.g., a 1 cm blemish is small or not.
- If 7 out of 10 experts say "small", we assign a degree $7/10 = 0.7$ to the statement "a 1 cm blemish is small".
- In general, if m out of n experts agree with the statement, we assign it a degree of certainty m/n.
- When we only have one expert, we cannot use polling.
- We can ask the expert to mark the degree of certainty in this statement on a scale, e.g., from 0 to 10.
- Such scales are known as *Likert scales*.
- If the expert selects 7 on a scale from 0 to 10, we assign, to this statement, a degree $7/10 = 0.7$.
- If the expert marks m on a scale from 0 to n, we assign a degree of certainty m/n.

2. Problem

- Both above elicitation methods are reasonable, both lead to reasonable useful results.
- However, usually, these two methods led to different membership degrees.
- It is therefore reasonable to find out how these different degrees are connected.
- Of course, degrees are subjective.
- In general, different experts assign different Likert-scale degrees of certainty to the same statement.
- We thus cannot expect an exact one-to-one correspondence between the polling and Likert-scale degrees.
- What we want to discover is an *approximate* relation between the corresponding scales.

3. Probabilistic Description of Polling Uncertainty

- Our main objective in describing the expert's knowledge is to use it.
- For example, we want to know whether a 1 cm blemish is small or not because:
  - one cure is proposed for a small blemish,
  - another for a large one.
- A doctor on whose patient with a 1 cm blemish the small-blemish cure worked will vote "small".
- A doctor on whose patient it didn't work will vote "No".
- The polling ratio m/n is equal to the frequency with which the small-blemish cure cures a 1 cm blemish.

4. From Frequencies to a Likert Scale: Main Idea

- If on average, P-method works on a half on x-objects, it does not mean that we always get $\mu_P(x) = 1/2$.
- We may get $\mu_P(x) < 1/2$ or $\mu_P(x) > 1/2$.
- Usually, frequencies $0/N$ and $1/N$ may come from the same probability $p = p'$.
- Similarly, $0/N$ and $2/N$ may come from the same prob.
- Eventually, we reach m_1 for which $f_0 = 0$ and $f_1 = m_1/N$ cannot come from the same prob.
- By repeating this procedure, we get a sequence of distinguishable frequencies $f_0 < f_1 < f_2 < \ldots$
- This is exactly what a Likert scale is about:
  - we have a finite number of possible estimates, and
  - to each situation, we place into correspondence one of these estimates.

5. From Probabilities to a Likert Scale: Details

- The observed frequency is $f = p + \Delta p$, where $E[\Delta p] = 0$ and $\sigma[\Delta p] = \sqrt{\dfrac{p(1-p)}{N}}$.
- If $p = p'$ for two frequenices $f \neq f'$, then $f - f' = \Delta p - \Delta p'$, with $\sigma[f - f'] = \sqrt{\dfrac{2p(1-p)}{N}}$.
- In statistics, such value is guaranteed to be different from 0 if $|f - f'| \geq k_0 \cdot \sigma$ (for $k_0 = 2, 3$, or 6).
- Thus, $f_{k+1} - f_k = k_0 \cdot \sqrt{\dfrac{2f_k(1-f_k)}{N}}$.
- For Likert-scale memb. f-n, $\mu(f_k) = \dfrac{k}{n}$, hence

$$\frac{1}{n} = \mu(f_{k+1}) - \mu(f_k) \approx \mu'(f_k) \cdot (f_{k+1} - f_k) = \mu'(f_k) \cdot k_0 \cdot \frac{2f_k(1-f_k)}{N}.$$

6. From Probabilities to a Likert Scale (cont-d)

$$\frac{1}{n} = \mu(f_{k+1}) - \mu(f_k) \approx \mu'(f_k)\cdot(f_{k+1} - f_k) = \mu'(f_k)\cdot k_0\cdot\frac{2f_k(1-f_k)}{N}.$$

- Thus, we get $\mu'(f) = \dfrac{c}{\sqrt{f(1-f)}}$.
- Solving this differential equation, we get $f = \sin^2(C\cdot\mu)$.
- The absolute confidence $\mu = 1$ corresponds to $f = 1$, hence
$$f \approx \sin^2\left(\frac{\pi}{2}\mu\right).$$
- At first glance, this relation looks very mathematical and non-intuitive.
- We will show that it becomes much clearer if we use the techniques of quantum computing.

7. Quantum Computing: Reminder

- In classical physics:
  - if we want to look for an element in an unsorted array of n elements,
  - then we need at least n computational steps.
- If we use fewer steps, we will not look into all n cells and thus, we may miss the desired element.
- In quantum case, we can perform the search in $\sqrt{n}$ steps (and $\sqrt{n} \ll n$).
- This possibility comes from the fact that in quantum physics:
  - in addition to the usual classical states,
  - we can also have *superpositions* of these states.

8. Superposition and Qubits

- For a *qubit* (quantum bit), superposition is a state $a_0\langle 0| + a_1\langle 1|$, where a_i are complex numbers.
- In quantum computing, only real values of a_0 and a_1 are used.
- Each such state can be described as a vector with coordinates (a_0, a_1) in a 2-D vector space.
- The probability p_i of observing i is equal to a_i^2.
- Since we always observe either 0 or 1, we must always have $p_0 + p_1 = a_0^2 + a_1^2 = 1$.
- In geometric terms, this means that the vector (a_0, a_1) must be on the unit circle with a center at 0.
- Each such vector is uniquely described by its angle φ with the $\langle 0|$-axis: $a_1 = \sin(\varphi)$, $a_0 = \cos(\varphi)$.

9. Resulting Relation Between Polling and Likert-Scale Degrees

- For each probability p, we can form a qubit state $\sqrt{p}\,\langle 1| + \sqrt{1-p}\,\langle 0|$ corresponding to this probability.
- For this state, $p = a_1^2 = \sin^2(\varphi)$.
- Due to the above relation between frequencies and Likert-scale values, we have $p \approx f \approx \sin^2\left(\frac{\pi}{2}\mu\right)$.
- Thus, we have $\sin^2(\varphi) \approx \sin^2\left(\frac{\pi}{2}\mu\right)$, hence

$$\varphi \approx \frac{\pi}{2}\mu.$$

- So, *the Likert-scale degree μ can be geometrically interpreted as (prop. to) the angle between the two states:*

$$\mu \approx \frac{2}{\pi}\varphi.$$

10. Superposition Between Two States

- Superposition is a basic operation in quantum physics:
  - in addition to superposition between the basic states $\langle 0|$ and $\langle 1|$,
  - we can also consider a superposition of states

$$\sqrt{p}\,\langle 1| + \sqrt{1-p}\,\langle 0| \text{ and } \sqrt{p'}\,\langle 1| + \sqrt{1-p'}\,\langle 0|.$$

- To describe a superposition, we:
  - add the corresponding vectors $(\sqrt{p}, \sqrt{1-p})$ and $(\sqrt{p'}, \sqrt{1-p'})$, and then
  - reduce the sum to the unit circle by dividing it by its length

$$\sqrt{(\sqrt{p}+\sqrt{p'})^2 + (\sqrt{1-p}+\sqrt{1-p'})^2}.$$

11. Fuzzy Interpretation of a Superposition Between Two States

- We consider superposition of qubit states

$$\sqrt{p}\,\langle 1| + \sqrt{1-p}\,\langle 0| \text{ and } \sqrt{p'}\,\langle 1| + \sqrt{1-p'}\,\langle 0|.$$

- In terms of probabilities, we get a complex expression:

$$p'' = \frac{(\sqrt{p}+\sqrt{p'})^2}{(\sqrt{p}+\sqrt{p'})^2 + (\sqrt{1-p}+\sqrt{1-p'})^2}.$$

- In terms of angles, $\varphi'' = \dfrac{\varphi + \varphi'}{2}$.
- Thus, $\mu'' = \dfrac{\mu + \mu'}{2}$.
- So, *superposition corresponds to simple averaging of Likert-scale degrees.*

12. Conclusion

- Two main techniques are used for eliciting a membership degree μ of a given statement S:
  - polling, when we ask n experts and take $\mu = m/n$ if m claim S to be true, we take $\mu = m/n$; and
  - a Likert-scale approach, when we take $\mu = m/n$ if an expert marks m on a 0 to n scale.
- Usually, these methods lead to different membership degrees.
- It is therefore reasonable to find out what is the relation between these two scales.
- To uncover such a relation, we analyze the meaning of both scales.
- In both cases, we need to estimate the degree $\mu_P(x)$ to which the value x satisfies the given fuzzy property P.

13. Conclusion (cont-d)

- We need to estimate the degree $\mu_P(x)$ to which the value x satisfies the given fuzzy property P.
- *Example:* the degree to which a 1 cm skin blemish is small; $P =$"small", $x =$1 cm.
- Classifying the blemish as small means we can apply techniques designed for small blemishes.
- From observations, we can find the probability p with which P-methods work for x-objects.
- An expert who observed that a P-method worked on an x-object will vote that x satisfies the property P.
- An expert who observed that a P-method did not work on an x-object will vote that x does not satisfy P.
- Thus, the polling membership degree is $f \approx p$.

14. Conclusion (cont-d)

- For a sample of limited size N, nearby frequencies $f \approx f'$ can come from the same probability.
- Only if the difference $f' - f$ is large enough, we can be sure that $p \neq p'$.
- We have frequencies 0, $1/N$, $2/N$, $\ldots$, $(N-1)/N$, 1, but much fewer *distinguishable* ones.
- It is therefore natural to associate these distinguishable probabilities with marks on a Likert scale.
- This leads to the relation $f \approx \sin^2\left(\frac{\pi}{2}\mu\right)$ between the polling memb. value f and the Likert-scale value μ.
- This relation is somewhat too mathematical and not very intuitively clear.

15. Conclusion (cont-d)

- It turns out that the relation becomes much clearer if we use models from quantum computing.
- An event with probability p is associated with a state $a_0\langle 0| + a_1\langle 1|$ s.t. $\text{Prob}(1) = p$.
- Then, $\mu \approx \dfrac{2}{\pi}\varphi$, where φ is an angle between the states $a_0\langle 0| + a_1\langle 1|$ and $\langle 0|$ ("false").
- Thus, quantum computing clarifies the relation between the polling and Likert-scale membership degrees:
  - a polling membership degree corresponds to the *probability* of observing the property, while
  - a Likert-scale membership degree is prop. to the *angle* between the given state and the "false" state.

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