Aggregation Operations from Quantum Computing

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1. Fuzzy Logic and Quantum Computing

- In the traditional Boolean (2-valued) logic, every statement is either true (1) or false (0).
- Both fuzzy logic and quantum computing extend the usual 2-valued logic, to handle uncertainty.
- In both cases, we need to extend usual Boolean operations (“and”, ”or”, “not”, etc.).
- In fuzzy logic, many different extensions are possible; it is often not clear which extension to use.
- Quantum logic provides a unique way of extending Boolean operations to more general case.
- In this talk, we describe a correspondence between fuzzy logic and quantum computing.
- We use this correspondence to describe corresponding fuzzy aggregation operations.

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2. Why Quantum Computing

- The speed of all processes is limited by the speed of light c.
- To send a signal across a 30 cm laptop, we need at least 1 ns; this corresponds to only 1 Gflop.
- If we want to make computers faster, we need to make processing elements smaller.
- Already, each processing cell consists of a few dozen molecules.
- If we decrease the size further, we get to the level of individual atoms and molecules.
- On this level, physics is different, it is quantum physics.
- One of the properties of quantum physics is its probabilistic nature (example: radioactive decay).

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3. Why Quantum Computing (cont-d)

- At first glance, this interferes with our desire to make reproducible computations.
- However, scientists learned how to make lemonade out of this lemon.
- First main discovery: Grover's quantum search algorithm.
- To search for an object in an unsorted array of n elements, we need, in the worst case, at least n steps.
- Reason: if we use fewer steps, we do not cover all the elements, and thus, we may miss the desired object.
- In quantum physics, we can find an element in $\sqrt{n}$ steps.
- For a Terabyte database, we get a million times speedup.
- Main idea: we can use superposition of different searches.

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4. Why Quantum Computing (cont-d)

- Another discovery: Shor's cracking RSA coding.
- The RSA algorithm is behind most secure transactions.
- A person selects two large prime numbers p_1 and p_2, and advertises their product $n = p_1 \cdot p_2$.
- By using this open code n, anyone can encode their message.
- To decode this message, one needs to know the factors p_1 and p_2.
- Factoring a large integer is known to be a computationally difficult problem.
- It turns out that with quantum computers, we can factor fast and thus, read all encrypted messages.
- The situation is not so bad: there is also a quantum encryption which cannot be easily cracked.

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5. Quantum States: Case of a Single Qubit (= *Quantum Bit*)

- A *bit* is a system which has two possible states 0 and 1.
- In quantum physics, in addition to $\langle 0|$ and $\langle 1|$, we also have *superpositions*

$$\alpha_0\langle 0| + \alpha_1\langle 1|.$$

- For each state, as a result of measurement, we always get either 0 or 1.
- The probability of observing 0 is equal to $|\alpha_0|^2$, and the probability of observing 1 is equal to $|\alpha_1|^2$.
- The total probability should be equal to 1:

$$|\alpha_0|^2 + |\alpha_1|^2 = 1.$$

- In general, α_i re complex numbers.
- (In quantum computing, only real values α_i are used.)

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6. A Natural Relation with Fuzzy

- Traditional probability theory describes objective probabilities – frequencies of different events.
- Fuzzy logic describes subjective opinions, what probabilists call *subjective probabilities.*
- It is therefore reasonable to associate a fuzzy degree $f \in [0, 1]$ with subjective probability.
- In a quantum state $\alpha_0\langle 0| + \alpha_1\langle 1|$, the probability of "true" is $|\alpha_1|^2$.
- If we identify this value with f, we get $\alpha_1^2 = f$ and $\alpha_0^2 = 1 - \alpha_1^2 = 1 - f$.
- Thus, $\alpha_0 = \sqrt{1 - f}$, $\alpha_1 = \sqrt{f}$, and the fuzzy degree f is associated with a state

$$\sqrt{1 - f}\langle 0| + \sqrt{f}\langle 1|.$$

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7. Quantum States of Multi-Qubit Systems

- In classical physics, a 2-qubit system has 4 possible states: 00, 01, 10, and 11.
- In quantum physics, we can have a superposition:

$$\alpha_{00}\langle 00| + \alpha_{01}\langle 01| + \alpha_{10}\langle 10| + \alpha_{11}\langle 11|.$$

- Here, the probability of observing 00 is $|\alpha_{00}|^2$, etc., so that
$$|\alpha_{00}|^2 + |\alpha_{01}|^2 + |\alpha_{10}|^2 + |\alpha_{11}|^2 = 1.$$
- A system of two independent qubits $\psi = \alpha_0\langle 0| + \alpha_1\langle 1|$ and $\psi' = \alpha'_0\langle 0| + \alpha'_1\langle 1|$ is described by a *tensor product*
$$\psi \otimes \psi' = \alpha_0 \cdot \alpha'_0\langle 00| + \alpha_0 \cdot \alpha'_1\langle 01| + \alpha_1 \cdot \alpha'_0\langle 00| + \alpha_1 \cdot \alpha'_1\langle 11|.$$
- A similar description holds for 3-, 4-, ..., N-qubit systems.

8. Resulting Relation with Fuzzy

- We decided to associate a fuzzy degree f with a state
$$\sqrt{1-f}\langle 0| + \sqrt{f}\langle 1|.$$
- A membership function $f(x)$ is described by several fuzzy degrees $f(x_1), \ldots, f(x_n)$.
- It is reasonable to assume that these degrees are, in some reasonable sense, independent.
- Thus, we associate a membership function with a tensor product of the corresponding quantum states:
$$\otimes_{i=1}^{n}\left(\sqrt{1-f(x_i)}\langle 0| + \sqrt{f(x_i)}\langle 1|\right).$$
- For example, for $n=2$, we get
$$\sqrt{1-f(x_1)}\cdot\sqrt{1-f(x_2)}\langle 00| + \sqrt{1-f(x_1)}\cdot\sqrt{f(x_2)}\langle 01| +$$
$$\sqrt{f(x_1)}\cdot\sqrt{1-f(x_2)}\langle 10| + \sqrt{f(x_1)}\cdot\sqrt{f(x_2)}\langle 11|.$$

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9. Quantum Operations and Transformations

- Quantum transformations should preserve superposition, so they should be linear.
- Quantum transformations should preserve the requirement that the total probability is 1.
- Such transformations are called *unitary*.
- The consequence is that all quantum transformations are reversible.
- We cannot have a simple "and"-operation for which $f(0,0) = f(0,1) = 0$.
- As a result, a quantum implementation of a function $y = f(x_1, \ldots, x_n)$ requires an extra bit x_0:

$$U_f : \langle x_1, \ldots, x_n, x_0| \to \langle x_1, \ldots, x_n, y|,\ y \stackrel{\text{def}}{=} x_0 \oplus f(x_1, \ldots, x_n).$$

- Here, $\oplus$ is "xor": $0 \oplus 1 = 1 \oplus 0 = 1$, $0 \oplus 0 = 1 \oplus 1 = 0$. This U_f is reversible: $x_0 = y \oplus f(x_1, \ldots, x_n)$.

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10. First Example: Negation $f(x_1) = \neg x_1$

- In general, we have

$$U_f : \langle x_1, \ldots, x_n, x_0| \to \langle x_1, \ldots, x_n, x_0 \oplus f(x_1, \ldots, x_n)|.$$

- For negation, $U_f : \langle x_1, x_0| \to \langle x_1, x_0 \oplus \neg x_1|$.
- For $x_0 = 0$, we get $U_f\langle 00| = \langle 01|$, $U_f\langle 10| = \langle 10|$.
- By linearity, for $\psi = (\alpha_0\langle 0| + \alpha_1\langle 1|) \otimes \langle 0|$, we get

$$U_f(\psi) = \alpha_0\langle 01| + \alpha_1\langle 10|.$$

- When $\alpha_0 = \sqrt{1-f}$ and $\alpha_1 = \sqrt{f}$, we get

$$U_f(\psi) = \sqrt{1-f}\langle 01| + \sqrt{f}\langle 10|.$$

- Here, $\text{Prob}(y = 1) = (\sqrt{1-f})^2 = 1 - f$.
- Thus, quantum-motivated negation is $\neg f = 1 - f$.

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11. Second Example: Intersection $f(x_1, x_2) = x_1 \,\&\, x_2$

- For intersection, $U_f : \langle x_1, x_2, x_0| \to \langle x_1, x_2, x_0 \oplus (x_1 \,\&\, x_2)|$.
- For $x_0 = 0$, we get:

$$U_f\langle 000| = \langle 000|, U_f\langle 010| = \langle 010|,$$

$$U_f\langle 100| = \langle 100|, U_f\langle 110| = \langle 111|.$$

- We want to apply U_f to $\psi = \psi_1 \otimes \psi_2 \otimes \langle 0|$, where

$\psi_1 = \sqrt{1 - f_1}\langle 0| + \sqrt{f_1}\langle 1|$ and $\psi_2 = \sqrt{1 - f_2}\langle 0| + \sqrt{f_2}\langle 1|$.

- By linearity, we get

$$U_f(\psi) = \sqrt{1 - f_1} \cdot \sqrt{1 - f_2}\langle 000| + \sqrt{1 - f_1} \cdot \sqrt{f_2}\langle 010| +$$

$$\sqrt{f_1} \cdot \sqrt{1 - f_2}\langle 100| + \sqrt{f_1} \cdot \sqrt{f_2}\langle 111|.$$

- Here, $\text{Prob}(y = 1) = (\sqrt{f_1} \cdot \sqrt{f_2})^2 = f_1 \cdot f_2$.
- Thus, quantum-motivated intersection is $f = f_1 \cdot f_2$.

12. Third Example: Union $f(x_1, x_2) = x_1 \vee x_2$

- For union, $U_f : \langle x_1, x_2, x_0| \to \langle x_1, x_2, x_0 \oplus (x_1 \vee x_2)|$.
- For $x_0 = 0$, we get:

$$U_f\langle 000| = \langle 000|, U_f\langle 010| = \langle 011|,$$
$$U_f\langle 100| = \langle 101|, U_f\langle 110| = \langle 111|.$$

- We want to apply U_f to $\psi = \psi_1 \otimes \psi_2 \otimes \langle 0|$, where $\psi_1 = \sqrt{1 - f_1}\langle 0| + \sqrt{f_1}\langle 1|$ and $\psi_2 = \sqrt{1 - f_2}\langle 0| + \sqrt{f_2}\langle 1|$.
- By linearity, we get $U_f(\psi) = \sqrt{1 - f_1} \cdot \sqrt{1 - f_2}\langle 000| + \sqrt{1 - f_1} \cdot \sqrt{f_2}\langle 011| +$

$$\sqrt{f_1} \cdot \sqrt{1 - f_2}\langle 101| + \sqrt{f_1} \cdot \sqrt{f_2}\langle 111|.$$

- $\text{Prob}(y = 1) = (\sqrt{1 - f_1} \cdot \sqrt{f_2})^2 + (\sqrt{f_1} \cdot \sqrt{1 - f_2})^2 + (\sqrt{f_1} \cdot \sqrt{f_2})^2 = (1 - f_1) \cdot f_2 + f_1 \cdot (1 - f_2) + f_1 \cdot f_2 = f_2 - f_1 \cdot f_2 + f_1 - f_1 \cdot f_2 + f_1 \cdot f_2 = f_1 + f_2 - f_1 \cdot f_2$.
- Thus, quantum-motivated union is $f = f_1 + f_2 - f_1 \cdot f_2$.

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13. 4th Example: Exclusive "Or" $f(x_1, x_2) = x_1 \oplus x_2$

- For union, $U_f : \langle x_1, x_2, x_0| \to \langle x_1, x_2, x_0 \oplus (x_1 \oplus x_2)|$.
- For $x_0 = 0$, we get:

$$U_f\langle 000| = \langle 000|, U_f\langle 010| = \langle 011|,$$
$$U_f\langle 100| = \langle 101|, U_f\langle 110| = \langle 110|.$$

- We want to apply U_f to $\psi = \psi_1 \otimes \psi_2 \otimes \langle 0|$, where $\psi_1 = \sqrt{1-f_1}\langle 0| + \sqrt{f_1}\langle 1|$ and $\psi_2 = \sqrt{1-f_2}\langle 0| + \sqrt{f_2}\langle 1|$.
- By linearity, we get

$$U_f(\psi) = \sqrt{1-f_1}\cdot\sqrt{1-f_2}\langle 000| + \sqrt{1-f_1}\cdot\sqrt{f_2}\langle 011| +$$
$$\sqrt{f_1}\cdot\sqrt{1-f_2}\langle 101| + \sqrt{f_1}\cdot\sqrt{f_2}\langle 110|.$$

- $\text{Prob}(y=1) = (\sqrt{1-f_1}\cdot\sqrt{f_2})^2 + (\sqrt{f_1}\cdot\sqrt{1-f_2})^2 = (1-f_1)\cdot f_2 + f_1\cdot(1-f_2) = f_2 - f_1\cdot f_2 + f_1 - f_1\cdot f_2 = f_1 + f_2 - 2f_1\cdot f_2$.
- Thus, quantum-motivated xor is $f = f_1 + f_2 - 2f_1\cdot f_2$.

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14. 5th Example: Set Difference $f(x_1, x_2) = x_1 - x_2$

- For union, $U_f : \langle x_1, x_2, x_0| \to \langle x_1, x_2, x_0 \oplus (x_1 - x_2)|$.
- For $x_0 = 0$, we get:

$$U_f\langle 000| = \langle 000|, U_f\langle 010| = \langle 010|,$$
$$U_f\langle 100| = \langle 101|, U_f\langle 110| = \langle 110|.$$

- We want to apply U_f to $\psi = \psi_1 \otimes \psi_2 \otimes \langle 0|$, where $\psi_1 = \sqrt{1 - f_1}\langle 0| + \sqrt{f_1}\langle 1|$ and $\psi_2 = \sqrt{1 - f_2}\langle 0| + \sqrt{f_2}\langle 1|$.
- By linearity, we get $U_f(\psi) = \sqrt{1 - f_1} \cdot \sqrt{1 - f_2}\langle 000| + \sqrt{1 - f_1} \cdot \sqrt{f_2}\langle 010| +$
$$\sqrt{f_1} \cdot \sqrt{1 - f_2}\langle 101| + \sqrt{f_1} \cdot \sqrt{f_2}\langle 110|.$$
- $\text{Prob}(y = 1) = (\sqrt{f_1} \cdot \sqrt{1 - f_2})^2 = f_1 \cdot (1 - f_2)$.
- Thus, quantum-motivated set difference is $f = f_1 \cdot (1 - f_2)$.

15. Superposition Leads to New Fuzzy Operations

- So far, we considered states $\sum a_i\langle\varphi_i|$ (e.g., $a_0\langle 0|+a_1\langle 1|$) with $\|a\|^2 \stackrel{\text{def}}{=} \sum |a_i|^2 = 1$.
- States with $\|a\|^2 \neq 1$ can be normalized: $a_i \to \dfrac{a_i}{\|a\|}$.
- For every two states ψ_1 and ψ_2 and for each a and b, we form a superposition $a\psi_1 + b\psi_2$ and normalize it.
- In particular, we can do it for fuzzy-related states $\psi_1 = \sqrt{1-f_1}\langle 0|+\sqrt{f_1}\langle 1|$ and $\psi_2 = \sqrt{1-f_2}\langle 0|+\sqrt{f_2}\langle 1|$.
- In the resulting state, the probability of 1 is equal to
$$f = \frac{\left(a\sqrt{f_1}+b\sqrt{f_2}\right)^2}{\left(a\sqrt{f_1}+b\sqrt{f_2}\right)^2+\left(a\sqrt{1-f_1}+b\sqrt{1-f_2}\right)^2}.$$
- This is a new fuzzy operation. For $a = b$, if we take $f_1 = \sin^2(\alpha_1)$ and $f_2 = \sin^2(\alpha_2)$, then $f_1 = \sin^2\left(\dfrac{\alpha_1+\alpha_2}{2}\right)$.

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