How to Estimate Expected Shortfall When Probabilities Are Known with Interval or Fuzzy Uncertainty

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1. How to Gauge Risk

- Engineers estimate the largest strength s_0 of historic floods and other natural disasters.
- Then they design the buildings so that they can withstand such disasters.
- However, there is always a possibility that the disaster strength S exceeds s_0.
- *Examples:* hurricane Katrina, Fukushima, etc.
- We cannot guarantee that $S \le s_0$.
- So, we should at least require $p = \text{Prob}(S > s_0) \le p_0$ for some small p_0.
- E.g., for manned space flights, NASA used $p_0 = 10^{-3}$.
- For reliability of a cell in a computer memory, we need $p_0 \ll 10^{-9}$: else one of the cells will be always faulty.

2. How to Gauge Risk (cont-d)

- It is also desirable to know how much damage will come, on average, if the threshold x_0 is exceeded.
- For each possible value S of the corresponding disaster strength, we estimate the corresponding damage X.
- Let x_p denote the damage corresponding to s_0, then

$$S \geq x_0 \text{ if and only if } X \geq x_p.$$

- Thus, we need to know the *expected shortfall*

$$\mathrm{ES}_p \stackrel{\text{def}}{=} E[X \mid X \geq x_p].$$

- The values x_p and ES_p is how we gauge the risk.
- Similar two measures are used in finance to describe the risk that an investment would result in a big loss.

3. How to Estimate ES_p: Ideal Case

- In the ideal case, we know the probability distribution that describes possible values of the damage X.
- A distribution is usually described by its *cumulative distribution function* (cdf) $F(x) \stackrel{\text{def}}{=} \text{Prob}(X \le x)$.
- The probability p_0 of exceeding the threshold x_p is equal to $1 - F(x_p)$, so $F(x_p) = 1 - p_0 = p$.
- For each p, the value x_p for which $F(x_p) = p$ is known as the p-th *quantile*:
  - for $p = 0.5$, we get the median;
  - for $p = 0.25$ and $p = 0.75$, we get *quartiles*, etc.
- The conditional expectation can then be computed as the ratio $\text{ES}_p = \dfrac{\int_{x_p}^{\infty} x\, dF(x)}{1 - p}$.

4. In Practice, We Only Have Partial Information About the Probabilities

- In practice, we rarely know the exact values of all the probabilities:
  - instead of the exact values $F(x)$ corresponding to different values x,
  - we only know an *interval* $[\underline{F}(x), \overline{F}(x)]$ that contains the actual (unknown) value $F(x)$.
- Such interval-valued cdf is known as a *probability box* (*p-box*, for short).
- More generally:
  - we may have several intervals $[\underline{F}(x), \overline{F}(x)]$;
  - these intervals correspond to different degrees of certainty $\alpha \in [0, 1]$.
- So, $F(x)$ is a sequence of embedded intervals, i.e., in effect, a *fuzzy number*.

5. How to Gauge Risk Under Interval (p-Box) and Fuzzy Uncertainty?

- For different cdfs $F(x) \in [\underline{F}(x), \overline{F}(x)]$ within a p-box, we get different quantiles x_p:
  - the smallest value x_p corresponds to the largest values $\overline{F}(x)$ of the cdf; while
  - the largest value x_p corresponds to the smallest values $\underline{F}(x)$ of the cdf.
- Thus, possible values of the quantile x_p form an interval $[\underline{x}_p, \overline{x}_p]$ in which $\overline{F}(\underline{x}_p) = \underline{F}(\overline{x}_p) = p$.
- To handle the fuzzy case, we take into account that:
  - for all $y = f(x_1, \ldots, x_n)$ with fuzzy x_i,
  - the alpha-cut ${}^{\alpha}\mathbf{y} \stackrel{\text{def}}{=} \{y : \mu(y) \geq \alpha\}$ of the result is equal to the range

$$f({}^{\alpha}\mathbf{x}_1, \ldots, {}^{\alpha}\mathbf{x}_n) = \{f(x_1, \ldots, x_n) : x_1 \in {}^{\alpha}\mathbf{x}_1, \ldots, x_n \in {}^{\alpha}\mathbf{x}_n(\alpha)\}.$$

6. Need to Gauge Risk Under Interval (p-Box) and Fuzzy Uncertainty (cont-d)

- So, to find the α-cut of the quantile x_p, we can:
  - compute the interval $[\underline{x}_p, \overline{x}_p]$
  - when each $F(x)$ belongs to the corresponding α-cut of the fuzzy number $\mathbf{F}(x)$.
- This straightforward computation is possible since the dependence of x_p on $F(x)$ is monotonic.
- So, the largest values of x_p is attained for smallest $F(x)$, and vice versa.
- For ES_p, there is no such clear monotonicity.
- We thus need a new algorithm for estimating ES_p under interval and fuzzy uncertainty.

7. What We Do

- We provide efficient algorithms for computing ES_p under interval (p-box) and fuzzy uncertainty.
- From the algorithmic viewpoint:
  - the problem of computing the expected shortfall under fuzzy uncertainty
  - can be reduced to the case of interval (p-box) uncertainty.
- Thus, we only need an algorithm for the interval (p-box) uncertainty.

8. Algorithm: Case of Interval Uncertainty

- We are given a p-box $[\underline{F}(x), \overline{F}(x)]$ and a probability p.
- We want to find the range $[\underline{\mathrm{ES}}_p, \overline{\mathrm{ES}}_p]$ of possible values of ES_p when cdf $F(x)$ is in this p-box.
- First, we compute $\overline{\mathrm{ES}}_p$ as ES_p corresponding to $F(x) = \underline{F}(x)$, i.e., as the ratio:

$$\frac{1}{1-p} \cdot \int_{\overline{x}_p}^{\infty} x\, d\underline{F}(x), \ \text{ where } \overline{x}_p \stackrel{\text{def}}{=} (\underline{F})^{-1}(p).$$

- Then, we compute $\underline{\mathrm{ES}}_p$ as ES_p corresponding to $F(x) = \overline{F}(x)$, i.e., as the ratio:

$$\frac{1}{1-p} \cdot \int_{\underline{x}_p}^{\infty} x\, d\overline{F}(x), \ \text{ where } \underline{x}_p \stackrel{\text{def}}{=} (\overline{F})^{-1}(p).$$

9. Algorithm: Case of Fuzzy Uncertainty

- We have a fuzzy-valued cdf $\mathbf{F}(x)$, i.e., we have the α-cuts ${}^\alpha\mathbf{F}(x) = [{}^\alpha\underline{F}(x), {}^\alpha\overline{F}(x)]$.
- We are also given a probability p.
- We want to to compute the α-cuts ${}^\alpha\mathbf{ES}_p = [{}^\alpha\underline{\mathrm{ES}}_p, {}^\alpha\overline{\mathrm{ES}}_p]$ of the expected shortfall $\mathbf{ES}_p$.
- First, we compute ${}^\alpha\underline{\mathrm{ES}}_p$ as ES_p corresponding to ${}^\alpha\underline{F}(x)$, i.e., as the ratio

$$\frac{1}{1-p} \cdot \int_{{}^\alpha\overline{x}_p}^{\infty} x\, d\,{}^\alpha\underline{F}(x), \text{ where } {}^\alpha\overline{x}_p \stackrel{\text{def}}{=} ({}^\alpha\underline{F})^{-1}(p).$$

- Then, we compute ${}^\alpha\overline{\mathrm{ES}}_p$ as ES_p corresponding to ${}^\alpha\overline{F}(x)$, i.e., as the ratio

$$\frac{1}{1-p} \cdot \int_{{}^\alpha\underline{x}_p}^{\infty} x\, d\,{}^\alpha\overline{F}(x), \text{ where } {}^\alpha\underline{x}_p \stackrel{\text{def}}{=} ({}^\alpha\overline{F})^{-1}(p).$$

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11. Analysis of the Problem

- We have $\mathrm{ES}_p = \dfrac{1}{1-p} \cdot I$, where $I \stackrel{\text{def}}{=} \int_{x_p}^\infty x\,dF(x)$; so:
  - ES_p attains its smallest possible value $\underline{\mathrm{ES}}_p$ when I attains its smallest possible value $\underline{I}$;
  - ES_p attains its largest possible value $\overline{\mathrm{ES}}_p$ when I attains its largest possible value $\overline{I}$.
- The integral I has an infinite upper bound.
- This integral can be thus represented as a limit of integrals I_T with a finite upper bound T when $T \to \infty$:

$$I = \lim_{T\to\infty} I_T, \text{ where } I_T \stackrel{\text{def}}{=} \int_{x_p}^T x\,dF(x).$$

- Thus, for very large T, we have $I \approx I_T$.

12. Analysis of the Problem (cont-d)

- $I_T = \int_{x_p}^T x\,dF(x)$ can be integrated by part:

$$I_T = x \cdot F(x)|_{x_p}^T - \int_{x_p}^T F(x)\,dx =$$

$$T \cdot F(T) - x_p \cdot F(x_p) - \int_{x_p}^T F(x)\,dx.$$

- For large T, we have $F(T)$ practically equal to $\lim\limits_{T\to\infty} F(T) = 1$, so $T \cdot F(T) = T$.
- By definition of a quantile x_p, we have $F(x_p) = p$, so

$$I_T = T - x_p \cdot p - \int_{x_p}^T F(x)\,dx.$$

13. When Does the Expression $I_T = T - x_p \cdot p - \int_{x_p}^T F(x)\,dx$ Attain Its Largest Value?

- Let $F^{\max}(x)$ be a cdf for which I_T is the largest, and let $x_p^{\max}$ be the corresponding value of x_p.
- For fixed x_p, the integral I_T is a decreasing function of the values $F(x)$.
- Thus, I_T is the largest when all $F(x)$ are the smallest.
- We have two limitations on the values $F(x)$ for $x \geq x_p$:
  - $\underline{F}(x) \leq F(x) \leq \overline{F}(x)$ from a given p-box;
  - $F(x) \geq p$ from $F(x_p) = p$ and monotonicity of $F(x)$.
- These constraints $\underline{F}(x) \leq F(x) \leq \overline{F}(x)$ and $F(x) \geq p$ can be equivalently described by a single constraint

$$\max(\underline{F}(x), p) \leq F(x) \leq \overline{F}(x).$$

14. When I_T Attains Its Largest Value?

- Thus, the smallest possible values of $F(x)$ correspond to $F(x) = \max(\underline{F}(x), p)$.
- When $\underline{F}(x) \geq p$, we have $\max(\underline{F}(x), p) = \underline{F}(x)$ and hence $F(x) = \underline{F}(x)$.
- The equality $\underline{F}(x) = p$ is equivalent to $x = \overline{x}_p$, thus the condition $\underline{F}(x) \geq p$ is equivalent to $x \geq \overline{x}_p$.
- When $\underline{F}(x) < p$, i.e., when $x < \overline{x}_p$, then $F(x) = p$; so:

$$\int_{x_p^{\max}}^T F(x)\,dx = \int_{x_p^{\max}}^{\overline{x}_p} p\,dx + \int_{\overline{x}_p}^T \underline{F}(x)\,dx =$$

$$(\overline{x}_p - x_p^{\max}) \cdot p + \int_{\overline{x}_p}^T \underline{F}(x)\,dx, \text{ and}$$

$$I_T = T - x_p^{\max} \cdot p - (\overline{x}_p - x_p^{\max}) \cdot p - \int_{\overline{x}_p}^T \underline{F}(x)\,dx.$$

15. When I_T Attains Its Largest Value: Result

- We have shown that

$$I_T = T - x_p^{\max} \cdot p - (\overline{x}_p - x_p^{\max}) \cdot p - \int_{\overline{x}_p}^{T} \underline{F}(x)\,dx.$$

- The two terms $x_p^{\max} \cdot p$ and $(\overline{x}_p - x_p^{\max}) \cdot p$ can be easily combined into a single term $\overline{x}_p \cdot p$, so

$$I_T = T - \overline{x}_p \cdot p - \int_{\overline{x}_p}^{T} \underline{F}(x)\,dx.$$

- Here, $\overline{x}_p$ is the quantile corresponding to the lower end-point $\underline{F}(x)$ of the p-box.
- So, we can conclude that the above expression is the value of the I_T corresponding to $F(x) = \underline{F}(x)$.
- Thus, the largest value of the integral I_T – and hence, of ES_p – is attained when $F(x) = \underline{F}(x)$.

16. When I_T Attains Its Smallest Value?

- Let $x_p^{\min}$ be the value corresponding to the cdf $F^{\min}(x)$ for which this integral is the largest possible.
- This means, in particular, that:
  - among all cdfs $F(x)$ with the same value of the p-th quantile $x_p^{\min}$ (i.e., for which $F(x_p^{\min}) = p$),
  - this particular cdf $F^{\min}(x)$ leads to the smallest possible value of the integral I_T.
- I_T is a decreasing function of the values $F(x)$.
- Thus, this integral is the smallest when all the values $F(x)$ are the largest.
- Under the limitations $\max(\underline{F}(x), p) \le F(x) \le \overline{F}(x)$, the largest possible values are $F(x) = \overline{F}(x)$.
- Thus, the smallest value of the integral I_T – and hence, of ES_p – is attained when $F(x) = \overline{F}(x)$.