It Is Possible to Determine Exact Fuzzy Values Based on an Ordering of Interval-Valued or Set-Valued Fuzzy Degrees

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1. For Fuzzy Degrees, Order Is More Important Than Numerical Values

- Fuzzy techniques describe subjective expert opinions.
- Based on the same knowledge,
  - some experts will be more "optimistic" and mostly use values close to 1 on a scale from 0 to 1, while
  - other experts may be more "pessimistic" and mostly use values close to 0 on the same scale.
- As a result, in fuzzy logic, the actual numerical value of a fuzzy degree can be different depending on a scale.
- What is important – and scale-independent – is the *order* between different values.

2. Formulation of the Problem

- Often, intervals like $[0.5, 0.6]$ provide a more adequate description of the expert's opinion than exact degrees.
- Here also, re-scalings are possible, so what is important is order between the degrees.
- If we have only order between the intervals, can we,
  - based on this order,
  - reconstruct the original numerical values – i.e., the "degenerate" intervals $[a, a]$?
- In this paper, we show that such a reconstruction is indeed possible.
- Moreover, we show that it is possible under three different definitions of order between intervals.

3. First Ordering: Lattice (Component-Wise) Order

- If for some statement:
  - the expert's degree of confidence is represented by an interval $[a, b]$, and
  - then we increase the lower bound, to make the interval $[a', b]$ with $a' > a$,
  - we thus increase our degree of confidence in this statement.
- Similarly, if we increase b to $b' > b$, we thus increase our degree of confidence.
- We thus say that $[a', b']$ represents a larger (or same) degree of confidence than $[a, b]$ if $a' \ge a$ and $b' > b$:

$$[a, b] \le [a', b'] \Leftrightarrow (a \le a' \,\&\, b \le b').$$

4. Another Way to Describe the Lattice Order

- The lattice order between $I = [a, b]$ and $I' = [a', b']$ can be described in the following equivalent way:

$$\forall x \in I\, \exists x' \in I'\, (x \le x')\, \&\, \forall x' \in I'\, \exists x \in I\, (x \le x'). \qquad (1)$$

- Indeed, if (1) is true, then:
  - since the first part of it is true for all $x \in I = [a, b]$,
  - in particular, it is true for $x = b$.
- Thus, for $x = b$, there exists a value $x' \in I' = [a', b']$ for which $b = x \le x'$.
- From $b \le x'$ and $x' \le b'$, we conclude that $b \le b'$.
- Similarly, from the second part of (1), for $x' = a'$, we conclude that $x \le x' = a'$ for some $x \in [a, b]$.
- From $a \le x$ and $x \le a'$, we conclude that $a \le a'$.
- Thus, (1) implies lattice order.

5. Proof (cont-d)

- Let us now prove that, vice versa, of $a \le a'$ and $b \le b'$, then

$$\forall x \in I\, \exists x' \in I'\, (x \le x')\, \&\, \forall x' \in I'\, \exists x \in I\, (x \le x').$$

- Indeed, if $I \le I'$ in the sense of the component-wise order, then:
  - we can take $x' = b'$ in the first part of the property and
  - we can take $b = x$ for the second part.
- The equivalence is proven.

6. Formulation of the Problem in Precise Terms

- Suppose that on the set of all subintervals $[a, b]$ of the interval $[0, 1]$, we have the lattice ordering.
- We will show that, based on this ordering, we can uniquely determine degenerate intervals $[a, a]$.
- This determination will be done step by step.
- First, we define $[0, 0]$ as the only interval I which is smaller than any other interval: $\forall J\,(I \le J)$.
- Then, we define intervals $[0, a]$ as such I for which the set of all J between $[0, 0]$ and I is linearly ordered:

$$\forall J\,\forall J'\,(([0,0] \le J \le I\,\&\,[0,0] \le J' \le I) \Rightarrow$$

$$(J \le J' \vee J' \le J)).$$

- Indeed, for $I = [0, a]$, all intermediate intervals J have the form $[0, c]$ for some c and are, thus, linearly ordered.

7. Determining Exact Fuzzy Values Based on the Ordering of Interval-Valued Degrees (cont-d)

- Vice versa, if $I = [a, b]$ with $a > 0$, then $J = [a, a]$ & $J' = [0, b]$ are between $[0, 0]$ and I but

$$J \not\leq J' \text{ and } J' \not\leq J.$$

- Finally, I is a degenerate interval $[a, a]$, with $a > 0$ if and only if
  - I is *not* of the type $[0, a]$ and
  - there exists an interval I' of the type $[0, a]$ for which
    * $I' \leq I$,
    * the set of all J between I' and I is linearly ordered, and
    * for no larger $I'' \geq I$, $I'' \neq I$, the set of all J between I' and I'' is linearly ordered.

8. Second Ordering: Necessarily Larger

- An interval $[a, b]$ means the actual (unknown) expert's degree of confidence $d(S)$ can be any value from $[a, b]$.
- An interval $[a', b']$ means that the actual expert's degree of confidence $d(S')$ can be any value from $[a', b']$.
- When we can be absolutely certain that $d(S) \leq d(S')$?
- The only way to be absolutely certain is to make sure that every $d(S) \in [a, b]$ is $\leq$ than every $d(S') \in [a', b']$.
- One can easily check that this is equivalent to

$$[a, b] \leq [a', b'] \Leftrightarrow b \leq a'.$$

- Based on this order, can we determine degenerate intervals $[a, a]$?

9. Second Ordering: Solution

- We show that $I \subseteq I' \Leftrightarrow$

$$\forall I'' \, ((I' \le I'' \Rightarrow I \le I'') \,\&\, (I'' \le I' \Rightarrow I'' \le I)) \quad (1).$$

- If $I \subseteq I'$, then (1) is clearly true.
- Vice versa, if (1) holds for $I = [a, b]$ and $I' = [a', b']$, then:
  - for $I'' = [b', b']$, we have $I' \le I''$;
  - thus, by (1), we have $I \le I''$, i.e., $[a, b] \le [b', b']$ and $b \le b'$.
- Similarly, we can prove that $a' \le a$, so $[a, b] \subseteq [a', b']$.
- Then, degenerate intervals are the ones that do not have any non-trivial subintervals.

10. Third Ordering: Possibly Larger

- *Idea:* some value $d(S) \in [a, b]$ is smaller than or equal than some value $d(S') \in [a', b']$.
- One can show that $[a, b] \le [a', b'] \Leftrightarrow a \le b'$.
- Let us show that $I \subseteq I' \Leftrightarrow$

$$\forall I'' \, ((I \le I'' \Rightarrow I' \le I'') \,\&\, (I'' \le I \Rightarrow I'' \le I')). \quad (2)$$

- Then, as before, we can determine degenerate intervals.
- If $I \subseteq I'$, then (2) is clearly true.
- Vice versa, if (2) holds for $I = [a, b]$ and $I' = [a', b']$:
  - then for $I'' = [a, a]$, we have $I'' \le I$;
  - thus, by (2), we have $I' \le I''$, i.e., $[a', b'] \le [a, a]$ and $a' \le a$.
- Similarly, we can prove that $b \le b'$, so $[a, b] \subseteq [a', b']$.

11. Set-Valued Degrees

- In some cases, not all the values from the interval $[\underline{\mu}(x), \overline{\mu}(x)]$ are possible.
- In such cases, instead of an *interval* of possible degrees, we have a *set* of degrees.
- It is reasonable to require that this set be closed, i.e., that if $x_n \in S$ and $x_n \to x$, then we should have $x \in S$.
- Indeed, in this case:
  - no matter how accurately we measure the degrees,
  - the limit value x will be – within this accuracy – indistinguishable from some x_n.
- Thus, we won't be able to conclude that x is not possible.
- So, it is reasonable to consider x to be possible.
- Set-valued fuzzy sets are also known as *hesitant sets*.

12. Set-Valued Degrees (cont-d)

- For sets, we similarly define the above three orders:
  - the "lattice-type"

$$(\forall s \in S\, \exists s' \in S'\, (s \le s'))\, \&\, (\forall s' \in S'\, \exists s \in S\, (s \le s'));$$

  - "necessarily larger" $\forall s \in S\, \forall s' \in S'\, (s \le s')$,
  - "possibly larger" $\exists s \in S\, \exists s' \in S'\, (s \le s')$.
- We prove that, based only on each of these three orders, we can determine one-point sets.
- Our proof is based on the lemma that, in all three orders, $S \le S' \Leftrightarrow [S] \le [S']$.
- Here, $[S]$ denotes the smallest interval containing S.

13. Set-Valued Degrees: Proof That $S \le S' \Leftrightarrow [S] \le [S']$

- Each of the sets S and S' is a subset of the bounded interval $[0,1]$ and is thus, bounded itself.
- For S, its infimum (greatest lower bound) a and its supremum (least upper bound) b are both finite.
- Both the infimum and the supremum are limits of elements from the set S.
- Since the set S is closed, these limits belong to S.
- So, we have $a \in S$, $b \in S$, and $a \le s \le b$ for all $s \in S$.
- Thus, the interval hull $[S]$ of S coincides with the interval $[a,b]$, where $a = \inf S$ and $b = \sup S$.
- Similarly, $[S'] = [a', b']$, where $a' = \inf S'$ and $b' = \sup S'$.

14. From Intervals (Pairs) to General Tuples

- In the traditional $[0,1]$-based fuzzy logic, the expert's degree of certainty is described by a number $a \in [0,1]$.
- In the interval-valued technique, the expert's degree of certainty is described by an interval $[a,b] \subseteq [0,1]$.
- Describing an interval is equivalent to describing a pair (a,b) of degrees, a pair for which $a \le b$.
- The most natural order between the intervals is a component-wise order:

$$(a,b) \le (a',b') \Leftrightarrow (a \le a' \text{ and } b \le b'.$$

- It's natural to consider n-element tuples $(a,b,\ldots,c)$ for which $a \le b \le \ldots \le c$ and

$$(a,\ldots,b) \le (a',\ldots,b') \Leftrightarrow (a \le a' \,\&\, \ldots \,\&\, b \le b').$$

15. Examples of n-Tuples

- Sometimes, we have an interval $[a, b]$ whose endpoints are only known with interval uncertainty, i.e.:
  - when we know the interval $[\underline{a}, \overline{a}]$ of possible values of the lower endpoint a and
  - we know the interval $[\underline{b}, \overline{b}]$ of possible values of the upper endpoint b.
- For such a *twin interval*, we have $\underline{a} \le \overline{a} \le \underline{b} \le \overline{b}$, i.e., a 4-tuple.
- In principle, we can consider cases in which $\underline{a}$, $\overline{a}$, $\underline{b}$, and $\overline{b}$ are also known with interval uncertainty.
- In this case, we have an n-tuple with $n = 8$.
- This constructions can be continued, so we n-tuples with larger and larger n.

16. Another Example: Interval-Valued Intuitionistic Fuzzy Degrees

- In intuitionistic fuzzy logic, expert's confidence in a statement S is described by 2 numbers:
  - a^+ is the expert's degree of certainty in S,
  - a^- is the expert's degree of certainty in $\neg S$.
- These values satisfy a natural inequality $a^+ \le 1 - a^-$.
- This description can be viewed as equivalent to considering an interval $[a^+, 1 - a^-]$ of possible values.
- We can consider interval-valued degrees, in which a^+ and a^- are known with interval uncertainty.
- This is $\Leftrightarrow$ interval $[a^+, 1 - a^-]$ in which endpoints are known with interval uncertainty – i.e., to a 4-tuple.

17. For Each n, We Can Reconstruct Fuzzy Values, i.e., Tuples $(a, a, , \ldots, a)$

- $0 = (0, 0, \ldots, 0)$ is the only degree for which $0 \le A$ for all A.
- The only A for which $\{B : 0 \le B \le A\}$ is linearly ordered are $A = (0, \ldots, 0, c)$ for some $c \ge 0$.
- After that, we can define degrees of the type $(0, \ldots, 0, b, c)$ with $0 < b$ as the degrees A for which:
  - the degree A is *not* of the type $(0, \ldots, 0, c)$, and
  - there exists a degree B of the type $(0, \ldots, 0, c)$ for which the set $\{C : B \le C \le A\}$ is linearly ordered.
- For a given B of the type $(0, \ldots, 0, c)$, we have several degrees A that satisfy these two properties.
- The largest of them is $(0, \ldots, 0, c, c)$.

18. Proof (cont-d)

- We can then define degrees $(0, \ldots, 0, b, c, c)$ with $0 < b$ as the degrees A for which:
  - the degree A is *not* of the type $(0, \ldots, 0, c, c)$, and
  - there exists a degree B of the type $(0, \ldots, 0, c, c)$ for which the set $\{C : B \le C \le A\}$ is linearly ordered.
- For a given B of the type $(0, \ldots, 0, c, c)$, we can have several degrees A that satisfy these two properties.
- The largest of them is $(0, \ldots, 0, c, c, c)$.
- Continuing along the same lines, we eventually arrive at the description of the degrees $(c, \ldots, c)$.
- The desired possibility to determine exact fuzzy values based on ordering between the n-tuples is thus proven.

19. Conclusions

- In many practical situations, it is useful to use generalizations of type-1 fuzzy, e.g., intervals.
- A natural question is then:
  - if we only have order between intervals,
  - can we use this order to determine degenerate intervals – which correspond to exact fuzzy values?
- We show that such a determination is always possible.
- We also show that this result can be extended beyond interval-valued fuzzy degrees – e.g., to set-valued ones.

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21. Appendix: Proofs, Starting with Lattice Order Case

- Suppose that on the set of all subintervals $[a, b]$ of the interval $[0, 1]$, we have the lattice ordering.
- We will show that, based on this ordering, we can uniquely determine degenerate intervals $[a, a]$.
- This determination will be done step by step.
- First, we define $[0, 0]$ as the only interval I which is smaller than any other interval: $\forall J\,(I \le J)$.
- Then, we define intervals $[0, a]$ as such I for which the set of all J between $[0, 0]$ and I is linearly ordered:

$$\forall J\,\forall J'\,(([0,0] \le J \le I\,\&\,[0,0] \le J' \le I) \Rightarrow (J \le J' \vee J' \le J)).$$

- Indeed, for $I = [0, a]$, all intermediate intervals J have the form $[0, c]$ for some c and are, thus, linearly ordered.

22. Determining Exact Fuzzy Values Based on the Ordering of Interval-Valued Degrees (cont-d)

- Vice versa, if $I = [a, b]$ with $a > 0$, then $J = [a, 0]$ & $J' = [0, b]$ are between $[0, 0]$ and I but

$$J \not\leq J' \text{ and } J' \not\leq J.$$

- Finally, I is a degenerate interval $[a, a]$, with $a > 0$ if and only if
  - I is *not* of the type $[0, a]$ and
  - there exists an interval I' of the type $[0, a]$ for which
    * $I' \leq I$,
    * the set of all J between I' and I is linearly ordered, and
    * for no larger $I'' \geq I$, $I'' \neq I$, the set of all J between I' and I'' is linearly ordered.

23. Proof of the First Result

- We want to prove that I is a degenerate interval $[a, a]$, with $a > 0$ if and only if
  - I is *not* of the type $[0, a]$ and
  - there exists an interval I' of the type $[0, a]$ for which
    * $I' \leq I$,
    * the set of all J between I' and I is linearly ordered, and
    * for no larger $I'' \geq I$, $I'' \neq I$, the set of all J between I' and I'' is linearly ordered.
- Indeed, if $I = [a, a]$ for some $a > 0$, then we can take $I' = [0, a]$ for this same a.
- Then all intervals J and J' between I' and I have the form $[b, a]$ for some b and the same a.
- They are, thus, are, linearly ordered.

24. Proof (cont-d)

- On the other hand, if $I'' = [a'', b'']$ is larger than $I = [a, a]$, this means that:
  - either $a'' > a$ – in which case $b'' \geq a'' > a$ and thus $b'' > a$
  - or $b'' > a$.
- Then, both $J = I$ and $J' = [0, b'']$ are between I' and I'', but $J \not\leq J'$ and $J' \not\leq J$.
- Vice versa, let us assume that I is a non-degenerate interval $[a_0, b_0]$ for some $a_0 < b_0$.
- The fact that this is not an interval of type $[0, a]$ means that $a_0 > 0$.
- In this case, linear ordering for all J and J' between $I' = [0, a]$ and $I = [a_0, b_0]$ is only possible if $b_0 = a$.
- Indeed, if $b_0 = a$, then we do get the linear ordering.

25. Proof (final)

- However, if $b_0 > a$, then $J = [0, b_0]$ and $J' = [a_0, a]$ are between I' and I, but $J \not\leq J'$ and $J' \leq J$.
- So, if there is a linear ordering of all intervals between I' and I, then:
  - we have $b_0 = a$, and
  - the interval I has the form $[a_0, a]$, with $a_0 < a$.
- However, now:
  - we can take a larger interval $I'' = [a, a] \geq I$, and
  - still be able to conclude that all intervals between I' and I'' are linearly ordered.
- This contradicts to our requirement that no such larger interval is possible.
- Thus, the above condition indeed uniquely determines degenerate intervals.

26. Proof That $S \le S' \Leftrightarrow [S] = [a,b] \le [S'] = [a',b']$: Case of "Lattice-Type" Order

- Let us first assume that $\forall s \in S\, \exists s' \in S', (s \le s')$.
- Let us prove that in this case, $\forall p \in [a,b]\, \exists p' \in [a',b']\, (p \le p')$.
- Indeed, $b = \sup S \in S$, so $\exists s' \in S'\, (b \le s')$.
- Since $p \in [a,b]$, we have $p \le b$ and thus, $p \le s'$.
- Due to $S' \subseteq [S'] = [a',b']$, we conclude that $s' \in [a',b']$.
- So, $p' \ge p$ for $p = s'$.
- Similarly, we can prove that if $\forall s' \in S'\, \exists s \in S\, (s \le s')$, then the same property holds for $[S]$ and $[S']$.

27. Case of "Lattice-Type" Order (cont-d)

- Vice versa, let's assume $\forall p \in [a, b]\, \exists p' \in [a', b']\, (p \le p')$.
- Let us then prove that $\forall s \in S. \exists s' \in S'\, (s \le s')$.
- Indeed, since $S \subseteq [S] = [a, b]$, the property $s \in S$ implies that $s \in [a, b]$.
- Thus, due to our assumption, there exists a $p' \in [a', b']$ for which $s \le p'$.
- From $p' \in [a', b']$, it follows that $p' \le b'$, thus $s \le b'$.
- We have shown that $b' \in S'$, thus we can take b' as the desired point $s' \in S'$.
- Similarly, we prove that $\forall p' \in [a', b']\, \exists p \in [a, b]\, (p \le p')$ implies $\forall s' \in S'\, \exists s \in S\, (s \le s')$.
- The equivalence is proven.

28. Case of "Necessarily Larger" Order

- Let us first prove that if $\forall x \in S\, \forall s' \in S'\, (s \le s')$, then $\forall p \in [a, b]\, \forall p' \in [a', b']\, (p \le p')$.
- Indeed, since $b \in S$ and $a' \in S'$, then, due to our assumption, we have $b \le a'$.
- Since $p \le b$ and $a' \le p'$, we thus have $p \le p'$.
- Vice versa, if $p \le p'$ for all $p \in [a, b]$ and $p' \in [a', b']$, then:
  - since $S \subseteq [a, b]$ and $S' \subseteq [a', b']$,
  - we have $s \le s'$ for all $s \in S$ and for all $s' \in S'$.
- The equivalence is proven.

29. Case of "Possibly Larger" Order

- If we have $s \le s'$ for some $s \in S$ and $s' \in S'$, then,
  - since $S \subseteq [a, b]$ and $S' \subseteq [a', b']$,
  - we have $s \le s'$ for $s \in [a, b]$ and $s' \in [a', b']$.
- Vice versa, let us assume that $p \le p'$ for some $p \in [a, b]$ and $p' \in [a', b']$.
- In this case, we have $s \le s'$ for some $s \in S$ and for some $s' \in S'$: namely, for $s = p$ and $s' = p'$.
- The equivalence is proven.