Decision Making Beyond Arrow's "Impossibility Theorem", with the Analysis of Effects of Collusion and Mutual Attraction

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(based on a joint work
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1. Group Decision Making and Arrow's Impossibility Theorem

- In 1951, Kenneth J. Arrow published his famous result about group decision making.
- This result that became one of the main reasons for his 1972 Nobel Prize.
- *The problem:*
  - A group of n participants $P_1, \ldots, P_n$ needs to select between one of m alternatives $A_1, \ldots, A_m$.
  - To find individual preferences, we ask each participant P_i to rank the alternatives A_j:
  $$A_{j_1} \succ_i A_{j_2} \succ_i \ldots \succ_i A_{j_n}.$$
  - Based on these n rankings, we must form a single group ranking (equivalence $\sim$ is allowed).

2. Case of Two Alternatives Is Easy

- *Simplest case:*
  - we have only two alternatives A_1 and A_2,
  - each participant either prefers A_1 or prefers A_2.
- *Solution:* it is reasonable, for a group:
  - to prefer A_1 if the majority prefers A_1,
  - to prefer A_2 if the majority prefers A_2, and
  - to claim A_1 and A_2 to be of equal quality for the group (denoted $A_1 \sim A_2$) if there is a tie.

3. Case of Three or More Alternatives Is Not Easy

- *Arrow's result:* no group decision rule can satisfy the following natural conditions.
- *Pareto condition:* if all participants prefer A_j to A_k, then the group should also prefer A_j to A_k.
- *Independence from Irrelevant Alternatives:* the group ranking of A_j vs. A_k should not depend on other A_is.
- *Arrow's theorem:* every group decision rule which satisfies these two condition is a *dictatorship* rule:
  - the group accepts the preferences of one of the participants as the group decision and
  - ignores the preferences of all other participants.
- This violates *symmetry*: that the group decision rules should not depend on the order of the participants.

4. Beyond Arrow's Impossibility Theorem

- *Usual claim:* Arrow's Impossibility Theorem proves that reasonable group decision making is impossible.
- *Our claim:* Arrow's result is only valid if we have binary ("yes"-"no") individual preferences.
- *Fact:* this information does not fully describe a persons' preferences.
- *Example:* the preference $A_1 \succ A_2 \succ A_3$:
  - it may indicate that a person strongly prefers A_1 to A_2, and strongly prefers A_2 to A_3, and
  - it may also indicate that this person strongly prefers A_1 to A_2, and at the same time, $A_2 \approx A_3$.
- *How can this distinction be described:* researchers in decision making use the notion of *utility*.

5. Why Utility

- *Idea of value:* a person's rational decisions are based on the relative values to the person of different outcomes.
- *Monetary value is often used:* in financial applications, the value is usually measured in monetary units (e.g., $).
- *Problem with monetary value:* the same monetary amount may have different values for different people:
  - a single dollar is likely to have more value to a poor person
  - than to a rich one.
- *Thus, a new scale is needed:* in view of this difference, in decision theory, researchers use a special *utility* scale.

6. What Is Utility: a Reminder

- *Main idea behind utility:* a common approach is based on preferences of a decision maker among *lotteries*.
- *Specifics:*
  - take a very undesirable outcome A^- and a very desirable outcome A^+;
  - consider the lottery $A(p)$ in which we get A^+ with given probability p and A^- with probability $1 - p$;
  - a utility $u(B)$ of an outcome B is defined as the probability p s.t. B is of the same quality as $A(p)$:

$$B \sim A(p) = A(u(B)).$$

- *Assumptions behind this definition:*
  - clearly, the larger p, the more preferable $A(p)$:

$$p < p' \Rightarrow A(p) < A(p');$$

  - the comparison amongst lotteries is a total order.

7. Different Utility Scales

- *Fact:* the numerical value $u(B)$ of the utility depends on the choice of A^- and A^+.
- *Natural question:* relate $u(B)$ with the values $u'(B)$ corr. to another choice of A^- and A^+.
- *Answer:* the utilities $u(B)$ and $u'(B)$ corresponding to different choices are related by a linear transformation:

$$u'(B) = a \cdot u(B) + b \text{ for some } a > 0 \text{ and } b.$$

- *Conclusion:* by using appropriate values a and b, we can re-scale utilities to make them more convenient.
- *Example:* in financial applications, we can make the scale closer to the monetary scale.

8. Problem

- *Situation:* we have n incompatible events $E_1, \ldots, E_n$ occurring with known probabilities $p_1, \ldots, p_n$.
- If E_i occurs, we get the outcome B_i.
- *Examples of events:*
  - coins can fall heads or tails;
  - dice can show 1 to 6.
- *We know:* the utility $u_i = u(B_i)$ of each outcome B_i.
- *Find:* the utility of the corresponding lottery.

9. Solution: Expected Utility

- *Main idea:* $u(B_i) = u_i$ means that B_i is equiv. to getting A^+ w/prob. u_i and A^- w/prob. $1 - u_i$.
- *Conclusion:* the lottery "B_i if E_i" is equivalent to the following two-step lottery:
  - first, we select E_i with probability p_i, and
  - then, for each i, we select A^+ with probability u_i and A^- with the probability $1 - u_i$.
- In this two-step lottery, the probability of getting A^+ is equal to

$$p_1 \cdot u_1 + \ldots + p_n \cdot u_n.$$

- *Result:* the utility of the lottery "if E_i then B_i" is

$$u = \sum_{i=1}^{n} p_i \cdot u_i = \sum_{i=1}^{n} p(E_i) \cdot u(B_i).$$

10. Nash's Bargaining Solution

- *How to describe preferences:* for each participant P_i, we can determine the utility $u_{ij} \stackrel{\text{def}}{=} u_i(A_j)$ of all A_j.
- *Question:* how to transform these utilities into a reasonable group decision rule?
- *Solution:* was provided by another future Nobelist John Nash.
- *Nash's assumptions:*
  - symmetry,
  - independence from irrelevant alternatives, and
  - *scale invariance* – under replacing function $u_i(A)$ with an equivalent function $a \cdot u_i(A)$,

11. Nash's Bargaining Solution (cont-d)

- *Nash's assumptions (reminder):*
  - symmetry,
  - independence from irrelevant alternatives, and
  - scale invariance.
- *Nash's result:*
  - the only group decision rule satisfying all these assumptions
  - is selecting an alternative A for which the product $\prod_{i=1}^{n} u_i(A)$ is the largest possible.
- *Comment.* the utility functions must be "scaled" s.t. the "status quo" situation $A^{(0)}$ has utility 0:

$$u_i(A) \to u_i'(A) \stackrel{\text{def}}{=} u_i(A) - u_i(A^{(0)}).$$

12. Properties of Nash's Solution

- *Nash's solution satisfies the Pareto condition:*
  - If all participants prefer A_j to A_k, this means that $u_i(A_j) > u_j(A_k)$ for every i,
  - hence $\prod_{i=1}^{n} u_i(A_j) > \prod_{i=1}^{n} u_i(A_k)$, which means that the group would prefer A_j to A_k.
- *Nash's solution satisfies the Independence condition:*
  - According to Nash's solution, we prefer A_j to A_k if $\prod_{i=1}^{n} u_i(A_j) > \prod_{i=1}^{n} u_i(A_k)$.
  - From this formula, once can easily see that
    * the group ranking between A_j and A_k
    * depends only on how participants rank A_j and A_k.

13. Comment: Nash's Solution Can Be Easily Explained in Terms of Fuzzy Logic

- We want all participants to be happy.
- So, we want the first participant to be happy *and* the second participant to be happy, etc.
- We can take:
  - $u_1(A)$ as the "degree of happiness" of the first participant,
  - $u_2(A)$ as the "degree of happiness" of the second participant, etc.
- To formalize "and", we use $d \cdot d'$ (one of the two "and"-operations originally proposed by L. Zadeh).
- Then, the degree to which all n participants are satisfied is equal to the product $u_1(A) \cdot u_2(A) \cdot \ldots \cdot u_n(A)$.

14. How We Can Determine Utility $u(B)$

- *General idea:* use the iterative *bisection* method.
- At every step, we have an interval $[\underline{u}, \overline{u}]$ containing the actual (unknown) value of the utility u.
- *Starting interval:* in the standard scale, $u \in [0, 1]$, so we can start with the interval $[\underline{u}, \overline{u}] = [0, 1]$.
- *Iteration:* once we have an interval $[\underline{u}, \overline{u}]$ that contains u, we:
  - compute its midpoint $u_{\text{mid}} \stackrel{\text{def}}{=} (\underline{u} + \overline{u})/2$, and
  - compare the alternative B with the lottery

$$A(u_{\text{mid}}) \stackrel{\text{def}}{=} \text{“}A^+ \text{ with probability } u_{\text{mid}}, \text{ otherwise } A^-\text{”}.$$

- *Possibilities:* $B \preceq A(u_{\text{mid}})$ and $A(u_{\text{mid}}) \preceq B$.

15. How We Can Determine Utility $u(B)$ (cont-d)

- *Reminder:* we know the values $\underline{u}$ and $\overline{u}$ such that $B \sim A(u)$ for some $u \in [\underline{u}, \overline{u}]$.
- *What we do:* we compute the midpoint $u_{\rm mid}$ of the interval $[\underline{u}, \overline{u}]$ and compare B with $L(u_{\rm mid})$.
- *Possibilities:* $B \preceq A(u_{\rm mid})$ and $A(u_{\rm mid}) \preceq B$.
- *Case 1:* if $B \preceq A(u_{\rm mid})$, then $u = u(B) \le u_{\rm mid}$, so
$$u \in [\underline{u}, u_{\rm mid}].$$
- *Case 2:* if $A(u_{\rm mid}) \preceq B$, then $u_{\rm mid} \le u = u(B)$, so
$$u \in [u_{\rm mid}, \overline{u}].$$
- After each iteration, we decrease the width of the interval $[\underline{u}, \overline{u}]$ by half.
- After k iterations, we get an interval of width 2^{-k} which contains the actual value u – i.e., u w/accuracy 2^{-k}.

16. Nash's Solution as a Way to Overcome Arrow's Paradox

- *Situation:* for each participant P_i ($i = 1, \ldots, n$), we know his/her utility $u_i(A_j)$ of A_j, $j = 1, \ldots, m$.
- *Possible choices:* lotteries $p = (p_1, \ldots, p_m)$ in which we select A_j with probability $p_j \geq 0$, $\sum\limits_{j=1}^{m} p_j = 1$.
- *Nash's solution:* among all the lotteries p, we select the one that maximizes

$$\prod_{i=1}^{n} u_i(p), \text{ where } u_i(p) = \sum_{j=1}^{m} p_j \cdot u_i(A_j).$$

- *Generic case:* no two vectors $u_i = (u_i(A_1), \ldots, u_i(A_m))$ are collinear.
- *In this general case:* Nash's solution is unique.

17. Sometimes It Is Beneficial to Cheat: An Example

- *Situation:* participant P_1 know the utilities of all the other participants, but they don't know his $u_1(B)$.
- *Notation:* let A_{m_1} be P_1's best alternative:

$$u_1(A_{m_1}) \geq u_1(A_j) \text{ for all } j \neq m_1.$$

- *How to cheat:* P_1 can force the group to select A_{m_1} by using a "fake" utility function $u_1'(A)$ for which
  - $u_1'(A_{m_1}) = 1$ and
  - $u_1'(A_j) = 0$ for all $j \neq m_1$.
- *Why it works:*
  - when selecting A_j w/$j \neq m_1$, we get $\prod u_i(A_j) = 0$;
  - when selecting A_{m_1}, we get $\prod u_i(A_j) > 0$.
- *This is a problem:* since Nash's solution depends on the assumption that we know the true preferences.

18. Cheating May Hurt the Cheater: an Observation

- *A more typical situation:* no one knows others' utility functions.
- Let P_1 use the above false utility function $u'_1(A)$ for which $u'_1(A_{m_1}) = 1$ and $u'_1(A_j) = 0$ for all $j \neq m_1$.
- *Possibility:* others use similar utilities with $u_i(A_{m_i}) > 0$ for some $m_i \neq m_1$ and $u_i(A_j) = 0$ for $j \neq m_i$.
- Then for every alternative A_j, Nash's product is equal to 0.
- From this viewpoint, all alternatives are equally good, so each of them can be chosen.
- In particular, it may be possible that the group selects an alternative A_q which is *the worst* for P_1 – i.e., s.t.

$$u_1(A_q) < u_1(A_j) \text{ for all } j \neq p.$$

19. Case Study: Territorial Division

- Dividing a set (territory) A between n participants, i.e., finding X_i s.t. $\bigcup\limits_{i=1}^{n} X_i$ and $X_i \cap X_j = \emptyset$ for $i \neq j$.
- The utility functions have the form $u_i(X) = \int_X v_i(t)\,dt$.
- *Nash's solution:* maximize $u_1(X) \cdot \ldots \cdot u_n(X_n)$.
- *Assumption:* P_1 does not know $u_i(B)$ for $i \neq 1$.
- *Choices:* the participant P_1 can report a fake utility function $v_1'(t) \neq v_1(t)$.
- For each $v_1'(t)$, we maximizes the product

$$\left(\int_{X_1} v_1'(t)\,dt\right) \cdot \left(\int_{X_2} v_2(t)\,dt\right) \cdot \ldots \cdot \left(\int_{X_n} v_n(t)\,dt\right).$$

- *Question:* select $v_1'(t)$ that maximizes the gain

$$u(v_1', v_1, v_2, \ldots, v_n) \stackrel{\text{def}}{=} \int_{X_1} v_1'(t)\,dt.$$

20. Decision Making under Uncertainty: a Reminder

- When deciding on v_1, the participant P_1 must make a *decision under uncertainty*.
- *Optimistic approach:* select A that maximizes the largest possible gain $u^+(A)$.
- *Pessimistic approach:* select A that maximizes the worst possible gain $u^-(A)$.
- Realistically, both approaches appear to be too extreme.
- *In real life:* it is more reasonable to use Hurwicz's pessimism-optimism criterion:
  - we choose a real number $\alpha \in [0, 1]$, and
  - choose an alternative A for which the combination

    $$u(A) = \alpha \cdot u^-(A) + (1 - \alpha) \cdot u^+(A)$$

    takes the largest possible value.

21. For Territorial Division, It Is Beneficial to Report the Correct Utilities: Result

- *Hurwicz's criterion* $u(A) = \alpha \cdot u^-(A) + (1-\alpha) \cdot u^+(A)$ may sound arbitrary.
- *Fact:* it can be deduced from scale- and shift-invariance.
- *For our problem:* Hurwicz's criterion means that we select a utility function $v_1'(t)$ that maximizes

$$J(v_1') \stackrel{\text{def}}{=} \alpha \cdot \min_{v_2,\ldots,v_n} u(v_1', v_1, v_2, \ldots, v_n) +$$

$$(1-\alpha) \cdot \max_{v_2,\ldots,v_n} u(v_1', v_1, v_2, \ldots, v_n).$$

- *Theorem:* when $\alpha > 0$, the objective function $J(v_1')$ attains its largest possible value for $v_1'(t) = v_1(t)$.
- *Conclusion:* unless we select pure optimism, it is best to select $v_1'(t) = v_1(t)$, i.e., to tell the truth.

22. How to Find Individual Preferences from Collective Decision Making: Inverse Problem of Game Theory

- *Situation:* we have a group of n participants $P_1, \ldots, P_n$ that does not want to reveal its individual preferences.
- *Example:* political groups tend to hide internal disagreements.
- *Objective:* detect individual preferences.
- *Example:* this is waht kremlinologies used to do.
- *Assumption:* the group uses Nash's solution to make decisions.
- *We can:* ask the group as a whole to compare different alternatives.

23. Comment

- *Fact:* Nash's solution depends only on the product of the utility functions.
- *Corollary:* in the best case,
  - we will be able to determine n individual utility functions
  - without knowing which of these functions corresponds to which individual.
- *Comment:* this is OK, because
  - our main objective is to predict future behavior of this group,
  - and in this prediction, it is irrelevant who has which utility function.

24. How to Find Individual Preferences from Collective Decision Making: Our Result

- Let $u_{ij} = u_i(A_j)$ denote i-th utility of j-th alternative.
- *We assume* that utility is normalized: $u_i(A_0) = 0$ for status quo A_0 and $u_i(A_1) = 1$ for some A_1.
- *According to Nash:* $p = (p_1, \ldots, p_n) \preceq q = (q_1, \ldots, q_n) \Leftrightarrow$

$$\prod_{i=1}^{n} \left(\sum_{j=1}^{n} p_j \cdot u_{ij} \right) \le \prod_{i=1}^{n} \left(\sum_{j=1}^{n} q_j \cdot u_{ij} \right).$$

- *Theorem:* if utilities u_{ij} and u'_{ij} lead to the same preference $\preceq$, then they differ only by permutation.
- *Conclusion:* we can determine individual preferences from group decisions.
- *An efficient algorithm* for determining u_{ij} from $\preceq$ is possible.

25. We Must Take Altruism and Love into Account

- *Implicit assumption:* the utility $u_i(A_j)$ of a participant P_i depends only on what he/she gains.
- *In real life:* the degree of a person's happiness also depends on the degree of happiness of other people:
  - Normally, this dependence is positive, i.e., we feel happier if other people are happy.
  - However, negative emotions such as jealousy are also common.
- This idea was developed by another future Nobelist Gary Becker: $u_i = u_i^{(0)} + \sum_{j \neq i} \alpha_{ij} \cdot u_j$, where:
  - $u_i^{(0)}$ is the utility of person i that does not take interdependence into account; and
  - u_j are utilities of other people $j \neq i$.

26. Paradox of Love

- *Case* $n = 2$*:* $u_1 = u_1^{(0)} + \alpha_{12} \cdot u_2$; $u_2 = u_2^{(0)} + \alpha_{21} \cdot u_1$.
- *Solution:* $u_1 = \dfrac{u_1^{(0)} + \alpha_{12} \cdot u_2^{(0)}}{1 - \alpha_{12} \cdot \alpha_{21}}$; $u_2 = \dfrac{u_2^{(0)} + \alpha_{21} \cdot u_1^{(0)}}{1 - \alpha_{12} \cdot \alpha_{21}}$.
- *Example:* mutual affection means that $\alpha_{12} > 0$ and $\alpha_{21} > 0$.
- *Example:* selfless love, when someone else's happiness means more than one's own, corresponds to $\alpha_{12} > 1$.
- *Paradox:*
  - when two people are deeply in love with each other ($\alpha_{12} > 1$ and $\alpha_{21} > 1$), then
  - positive original pleasures $u_i^{(0)} > 0$ lead to $u_i < 0$ – i.e., to unhappiness.

27. Paradox of Love: Discussion

- *Paradox – reminder:*
  - when two people are deeply in love with each other, then
  - positive original pleasures $u_i^{(0)} > 0$ lead to unhappiness.
- This may explain why people in love often experience deep negative emotions.
- From this viewpoint, a situation when
  - one person loves deeply and
  - another rather allows him- or herself to be loved

  may lead to more happiness than mutual passionate love.

28. Why Two and not Three?

- An *ideal love* is when each person treats other's emotions almost the same way as one's own, i.e.,

$$\alpha_{12} = \alpha_{21} = \alpha = 1 - \varepsilon \text{ for a small } \varepsilon > 0.$$

- For *two people*, from $u_i^{(0)} > 0$, we get $u_i > 0$ – i.e., we can still have happiness.
- For $n \geq 3$, even for $u_i^{(0)} = u^{(0)} > 0$, we get

$$u_i = \frac{u^{(0)}}{1 - (1 - \varepsilon) \cdot (n - 1)} < 0, \text{ i.e., unhappiness.}$$

- *Corollary:* if two people care about the same person (e.g., his mother and his wife),
    - all three of them are happier
    - if there is some negative feeling (e.g., jealousy) between them.

29. Emotional vs. Objective Interdependence

- *We considered:* emotional interdependence, when one's utility is determined by the utility of other people:

$$u_i = u_i^{(0)} + \sum_j \alpha_j \cdot u_j.$$

- *Alternative:* "objective" altruism, when one's utility depends on the material gain of other people:

$$u_i = u_i^{(0)} + \sum_j \alpha_j \cdot u_j^{(0)}.$$

- *In this approach:* we care about others' well-being, not about their emotions.
- *In this approach:* no paradoxes arise, any degree of altruism only improves the situation.
- The objective approach was proposed by yet another Nobel Prize winner Amartya K. Sen.

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