Why Intervals? Why Fuzzy Numbers? Towards a New Justification

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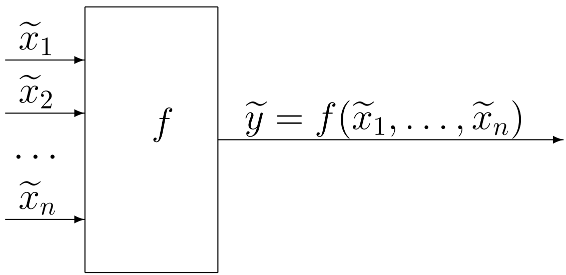
1. General Problem of Data Processing under Uncertainty

- *Indirect measurements:* way to measure y that are are difficult (or even impossible) to measure directly.
- *Idea:* $y = f(x_1, \ldots, x_n)$

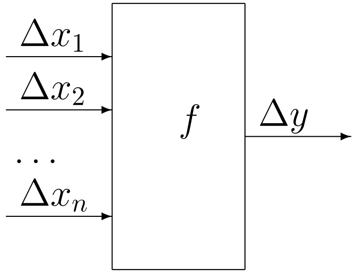


- *Problem:* measurements are never 100% accurate: $\widetilde{x}_i \neq x_i$ $(\Delta x_i \neq 0)$ hence

$$\widetilde{y} = f(\widetilde{x}_1, \ldots, \widetilde{x}_n) \neq y = f(x_1, \ldots, x_n).$$

- *Question:* what are bounds on $\Delta y \stackrel{\text{def}}{=} \widetilde{y} - y$?

2. Probabilistic and Interval Uncertainty



- *Traditional approach:* we know probability distribution for Δx_i (usually Gaussian).
- *Where it comes from:* calibration using standard MI.
- *Problem:* calibration is not possible in:
  - fundamental science
  - manufacturing
- *Solution:* we know upper bounds Δ_i on $|\Delta x_i|$ hence

$$x_i \in [\widetilde{x}_i - \Delta_i, \widetilde{x}_i + \Delta_i].$$

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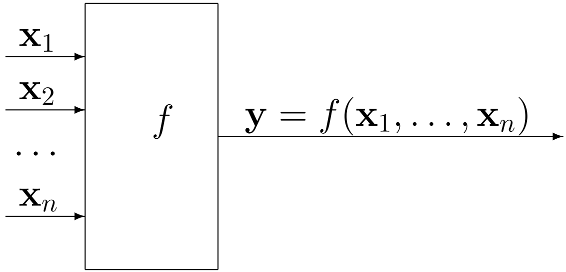
3. Interval Computations: A Problem



- *Given:* an algorithm $y = f(x_1, \ldots, x_n)$ and n intervals $\mathbf{x}_i = [\underline{x}_i, \overline{x}_i]$.
- *Compute:* the corresponding range of y:

$$[\underline{y}, \overline{y}] = \{f(x_1, \ldots, x_n) \mid x_1 \in [\underline{x}_1, \overline{x}_1], \ldots, x_n \in [\underline{x}_n, \overline{x}_n]\}.$$

- *Fact:* NP-hard even for quadratic f.
- *Challenge:* when are feasible algorithm possible?
- *Challenge:* when computing $\mathbf{y} = [\underline{y}, \overline{y}]$ is not feasible, find a good approximation $\mathbf{Y} \supseteq \mathbf{y}$.

4. Possibility of Linearization

- Situation: In practice, $|\Delta x_i| \ll |\widetilde{x}_i|$.
- *Possibility:* we can linearize the dependence of y on x_i:

  $$f(x_1, \ldots, x_n) = f(\widetilde{x}_1 - \Delta x_1, \ldots, \widetilde{x}_n - \Delta x_n) \approx \widetilde{y} - \Delta y,$$

  where

  $$\Delta y \stackrel{\text{def}}{=} c_1 \cdot \Delta x_1 + \ldots + c_n \cdot \Delta x_n,$$
  $$\widetilde{y} \stackrel{\text{def}}{=} f(\widetilde{x}_1, \ldots, \widetilde{x}_n), \quad c_i \stackrel{\text{def}}{=} \frac{\partial f}{\partial x_i}.$$

- *Towards resulting formula:* when $\Delta x_i \in [-\Delta_i, \Delta_i]$, then the largest possible value of the linear combination Δy is

  $$\Delta = |c_1| \cdot \Delta_1 + \ldots + |c_n| \cdot \Delta_n,$$

  and the smallest possible value of δy is $-\Delta$.

- *Resulting formula:* the interval of possible values of Δy is $[-\Delta, \Delta]$, and the desired interval of possible values of y is $[\widetilde{y} - \Delta, \widetilde{y} + \Delta]$.

5. Interval Uncertainty and Interval Computations: First Traditional Challenge – Non-Linearity

- *Usual case:* measurement errors Δx are small, so quadratic terms can be ignored.
- *Example:* if $\Delta x \approx 3\%$, then

$$(\Delta x)^2 \approx 0.03^2 \approx 0.1\% \ll 3\%.$$

- *Challenge:* in some practically important cases, non-linear terms cannot be ignored.
- *Example:* if $\Delta x \approx 30\%$, then $(\Delta x)^2$ is $\approx 10\%$ – no longer much smaller than Δx.
- *Difficulty:* NP-hard even for quadratic terms.
- *Successes:* researchers have *designed* several useful algorithms, and have successfully *used* these algorithms in numerous practical applications.

6. Second Traditional Challenge: Partial Information About Probabilities

- *So far:* we have described two extreme situations:
  - complete information about probabilities of values $x_i \in \mathbf{x}_i$ (probabilistic uncertainty);
  - no information about probabilities of values $x_i \in \mathbf{x}_i$ (interval uncertainty).
- *In practice:* we often have *partial* information about these probabilities.
- *How to represent this uncertainty:*
  - bounds $[\underline{F}(x), \overline{F}(x)]$ on the (unknown) cumulative distribution function $F(x)$ (such bounds are called *p-boxes*),
  - bounds on moments, etc.
- In many such situations, there are efficient algorithms and successful applications.

7. Interval Uncertainty and Interval Computations: A New Challenge

- *Typical situation:* $|\Delta x| \le \Delta$.
- *Conclusion:* the actual value x is in the interval $\mathbf{x} = [\widetilde{x} - \Delta, \widetilde{x} + \Delta]$.
- *Open question:* are all values from $\mathbf{x}$ possible?
- *Alternative formulation:* is the set X of all possible values of x equal to $\mathbf{x}$?
- *Clarification:* the bound Δ is often an overestimation, so literally $X \subset \mathbf{x}$.
- *Refined question:* is X an interval?
- *Related computational questions:* if X is not an interval, then:
  - how do we approximate such sets?
  - how do we process the resulting uncertainty?

8. What Is Known About This New Challenge: Description and Limitations

There are several results which justify the use of intervals:

- limit approach;
- consistency approach;
- invertibility approach.

9. Limit Approach

- *Fact:* the measurement error Δx is usually a sum of several small independent errors Δx_i: $\Delta x = \sum_{i=1}^{n} \Delta x_i$.
- *Known probabilistic result:* due to the Central Limit Theorem, the distribution for the sum is close to Gaussian.
- *Similar interval-related result:*
  - if each Δx_i independently takes values in some set X_i,
  - then the set of possible values for Δx is close to an interval.
- *Limitations:*
  - in practice, we sometimes have error components which are much larger than others;
  - in this case, the above approach does not work.

10. Consistency Approach

- *Idea:* for expert estimates, it is important to be able to check consistency.
- *For intervals:* $\cap X_i \neq \emptyset$ if and only if $X_i \cap X_j \neq \emptyset$.
- *Conclusion:* when we add a new piece of knowledge X_{n+1} to n consistent sets, it is sufficient to check n pairwise consistencies.
- *Theorem:* this only hold for intervals (or multi-D boxes).
- *Limitations:*
  - consistency is important for *expert* estimates;
  - for *measurement* uncertainty, consistency is automatic (since the actual value is guaranteed to be in all the sets X_i).
- *Conclusion:* consistency checking is *not* a good foundation for measurement uncertainty.

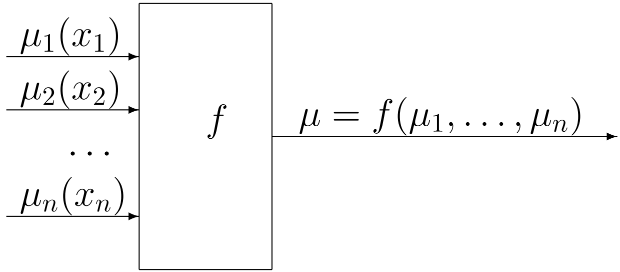
11. Invertibility Approach

- *For exact numbers:*
  - if we accidentally add a wrong number y to x,
  - we can undo it by subtracting y from the $x+y$.
- *For intervals:* we can also undo.
- *Caution:* $(\mathbf{x}+\mathbf{y})\ominus\mathbf{y}=\mathbf{x}$, but $(\mathbf{x}+\mathbf{y})-\mathbf{y}\neq\mathbf{x}$.
- *Theorem:* if we add any non-interval set to intervals, we lose invertibility.
- *Advantage:* in AI-type applications, we explore possible data processing algorithms, undo makes sense.
- *Limitations:* in data processing, the algorithm is usually well known and well established.
- *Conclusion:* this is *not* a good foundation for measurement uncertainty.

12. Summary

- *Conclusion:* the existing justifications of intervals are not fully helpful for measurement-related uncertainty.
- *Desirable:* it is therefore desirable to come up with a new more convincing justification.
- *Plan:* this is what we will do in this paper.

13. Fuzzy Computations: A Problem



- *Given:* an algorithm $y = f(x_1, \ldots, x_n)$ and n fuzzy numbers $\mu_i(x_i)$.
- *Compute:* $\mu(y) = \max\limits_{x_1,\ldots,x_n: f(x_1,\ldots,x_n)=y} \min(\mu_1(x_1), \ldots, \mu_n(x_n))$.
- *Motivation:* y is a possible value of $Y \leftrightarrow \exists x_1, \ldots, x_n$ s.t. each x_i is a possible value of X_i and $f(x_1, \ldots, x_n) = y$.
- *Details:* "and" is min, $\exists$ ("or") is max, hence

  $$\mu(y) = \max_{x_1,\ldots,x_n} \min(\mu_1(x_1), \ldots, \mu_n(x_n), t(f(x_1, \ldots, x_n) = y)),$$

  where $t(\text{true}) = 1$ and $t(\text{false}) = 0$.

14. Fuzzy Computations: Reduction to Set Computations

- *Problem (reminder):*
  - *Given:* an algorithm $y = f(x_1, \ldots, x_n)$ and n fuzzy numbers X_i described by membership functions $\mu_i(x_i)$.
  - *Compute:* $Y = f(X_1, \ldots, X_n)$, where Y is defined by Zadeh's extension principle:

$$\mu(y) = \max_{x_1,\ldots,x_n : f(x_1,\ldots,x_n)=y} \min(\mu_1(x_1), \ldots, \mu_n(x_n)).$$

- *Idea:* represent each X_i by its α-cuts

$$X_i(\alpha) = \{x_i : \mu_i(x_i) \geq \alpha\}.$$

- *Advantage:* for continuous f, for every α, we have

$$Y(\alpha) = f(X_1(\alpha), \ldots, X_n(\alpha)).$$

- *Resulting algorithm:* for $\alpha = 0, 0.1, 0.2, \ldots, 1$ apply interval computations techniques to compute $Y(\alpha)$.

15. Similar Open Problem

- *In principle:* we can use arbitrary fuzzy sets.
- *Sometimes:* there is a practical need for complex fuzzy sets.
- *Example:*
  - suppose that all we know about x is that $x^2 \in [1, 4]$;
  - then, the set of possible values of x is a non-interval set $[-2, -1] \cup [1, 2]$.
- *In practice:* mostly *fuzzy numbers* are used, i.e., fuzzy sets X for which all α-cuts $X(\alpha)$ are intervals.
- *Challenge:* we need to explain this empirical phenomenon.

16. Towards a New Justification of Intervals and Fuzzy Numbers

- *Reminder:* there are good justifications for the use of intervals in expert uncertainty.
- *Problem:* we are looking for measurement-related justification.
- *Objective:* describe a family $\mathcal{F}$ of sets X.
- *Bounded sets:* we are always guaranteed that X is contained in an interval $[\widetilde{x} - \Delta, \widetilde{x} + \Delta]$.
- *Conclusion:* all sets $X \in \mathcal{F}$ must be bounded.
- *Limit argument:* if values $x^{(1)}, x^{(2)}, \ldots$ are all possible values of x, and $x^{(n)} \to x_0$, then for every accuracy $x^{(n)} \approx x$ for sufficiently large n.
- *Conclusion:* from the practical viewpoint, limit values are also possible, so all sets $X \in \mathcal{F}$ are closed.

17. Towards a New Justification of Intervals (cont-d)

- *Similarly:* if X_i are possible and $X_i \to X$, then X should also be possible (i.e., the family $\mathcal{F}$ should be closed).
- *Need to fuse measurement results:*
  - after each of n measurements, we have $x \in X_i$, $1 \le i \le n$;
  - the resulting joint measurement leads to $\cap X_i$.
- *Conclusion:* $\mathcal{F}$ must be closed under intersection.
- *Need for data processing:* in the linearized case, $y = c_0 + \sum c_i \cdot x_i$.
- *Conclusion:* the set $\{c_0 + \sum c_i \cdot x_i : x_i \in X_i\}$ must be in $\mathcal{F}$.
- *Need for computer representation:* the family $\mathcal{F}$ must be finite-parametric.

18. Main Result

Theorem. *Let $\mathcal{F}$ be a non-empty closed finite-dimensional family of bounded closed sets of R which is:*

- *closed under intersection* $\bigcap\limits_{i=1}^{n} X_i$, *and*
- *closed under linear combination*

$$\left\{c_0 + \sum c_i \cdot x_i : x_i \in X_i\right\}.$$

Then, this family $\mathcal{F}$ is

- *either the family of all one-point sets*
- *or the family of all intervals.*

Comment: thus, if we exclude the case when all the values are known exactly, we get a new justification for intervals.

19. Proof: Part 1

- *Case to consider:* $\mathcal{F}$ contains only 1-point sets.
- *What we prove in this case:* that $\mathcal{F}$ contains all 1-point sets.
- *Proof:*
  - let $\{x_0\} \in \mathcal{F}$;
  - let $x \in R$ be an arbitrary real number;
  - then, for $c_0 = x - x_0$ and $c_1 = 1$, we have

$$\{c_0 + c_1 \cdot x_1 : x_1 \in X_1\} = \{(x - x_0) + x_0\} = \{x\};$$

  - since $\mathcal{F}$ is closed under linear combinations, we have $\{x\} \in \mathcal{F}$.
- *Conclusion:* in this case, $\mathcal{F}$ coincides with the family of all 1-point sets.

20. Proof: Part 2

- *Remaining case:* $\mathcal{F}$ contains a set X with at least two points.
- For this set, $\inf X < \sup X$.
- *What we plan to prove:* that $\mathcal{F}$ contains a set Y with $\inf Y = 0$ and $\sup Y = 1$.
- *Proof:* take $Y = c_0 + c_1 \cdot X$, with $c_0 = -\inf X$ and $c_1 = \dfrac{1}{\sup X - \inf X}$.
- *Additional statement:*
  - *we assumed:* that every set from $\mathcal{F}$ is closed;
  - *conclusion:* the set Y contains its own inf and sup, so $\{0, 1\} \subseteq Y$.

21. Proof: Part 3

- *We plan to prove:* that $\mathcal{F}$ contains the interval $[0,1]$.
- *We start with:* Y such that $\{0,1\} \subseteq Y \subseteq [0,1]$.
- *Conclusion:* for every m, $\mathcal{F}$ contains

$$Y_m \stackrel{\text{def}}{=} \frac{1}{m} \cdot Y + \ldots + \frac{1}{m} \cdot Y \quad (m \text{ times}).$$

- Every element of Y_m has the form $(y_1 + \ldots + y_m)/m$, where $y_i \in Y$.
- Since $\inf Y = 0$ and $\sup Y = 1$, every element from Y is between 0 and 1.
- Thus, every element from Y_m is also between 0 and 1: $Y_m \subseteq [0,1]$.
- By taking values $y_i = 0$ and 1, we conclude that $\{0, 1/m, 2/m, \ldots, 1\} \subseteq Y_m \subseteq [0,1]$.
- When $m \to \infty$, we get $[0,1] \in \mathcal{F}$.

22. Proof: Part 4

- *What we need to prove:* that $\mathcal{F}$ contains an arbitrary interval $[a, b]$, with $a \le b$.
- *We have already shown:* that $\mathcal{F}$ contains $[0, 1]$.
- *We know:* that $\mathcal{F}$ is closed under linear combination.
- *In particular:* $\mathcal{F}$ contains $a + (b - a) \cdot [0, 1]$.
- *Observation:* this is exactly the interval $[a, b]$.
- *Conclusion:* $\mathcal{F}$ contains all intervals.

23. Proof: Part 5

- *What we have proven:* that $\mathcal{F}$ contains all intervals.
- *What remains to prove:* that $\mathcal{F}$ has no other sets.
- *Reduction to a contradiction:* let us assume that $\mathcal{F}$ contains a non-interval set S.
- In other words, there exists $s_0 \in [\inf S, \sup S]$ for which $s_0 \notin S$.
- Let $s^+ \stackrel{\text{def}}{=} \inf\{s \in S : s > s_0\}$, $s^- \stackrel{\text{def}}{=} \sup\{s \in S : s < s_0\}$.
- Since S is a closed set, it contains both s^- and s^+.
- By definition of s^- and s^+, the set S cannot contain any elements from the open interval (s^-, s^+).
- Thus, $S \cap [s^-, s^+] = \{s^-, s^+\}$.
- Since the family $\mathcal{F}$ is closed under intersection, it contains $S \cap [s^-, s^+] = \{s^-, s^+\}$.

24. Proof: Final Part

- *We have proven:* that $\mathcal{F}$ contains a 2-point set $\{s^-, s^+\}$.
- *Conclusion:* for every $a < b$, $\mathcal{F}$ contains $\{a, b\} = c_0 + c \cdot \{s^-, s^+\}$, where $c_0 = a - c \cdot s^-$ and $c_1 = (b-a)/(s^+ - s^-)$.
- *In particular:* for every i, $\{0, 2^i\} \in \mathcal{F}$.
- *Conclusion:* for every $c_0, \ldots, c_n$, we have

$$C(c_0, \ldots, c_n) \stackrel{\text{def}}{=} c_0 \cdot \{0, 2^0\} + c_1 \cdot \{0, 2^1\} + \ldots + c_n \cdot \{0, 2^n\} \in \mathcal{F}.$$

- For $c_i \approx 1$, all these sets are different: each coefficient c_i can be determined from the value $c_i \cdot 2^i \in C(c_0, \ldots, c_n)$, and these values (for $c_i \approx 1$) are all different.
- Thus, we have a subfamily $C(c_0, \ldots, c_n)$ which is determined by $n + 1$ parameters $c_0, \ldots, c_n$.
- For $n > d$, this contradicts to our assumption that the family $\mathcal{F}$ is finite (d-) dimensional.

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