It Is Important to Take All Available Information into Account When Making a Decision: Case of the Two Envelopes Problem

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1. Traditional Decision Theory: Reminder

- Decision theory is based on the notion of *utility*.
- To describe this meaning, we need to select two alternatives:
  - a very bad alternative A_- which is worse that anything that we will actually encounter, and
  - a very good alternative A_+ which is better than anything that we will actually encounter.
- Then, for each number p from the interval $[0,1]$, we can form a lottery $L(p)$ in which:
  - we get A_+ with probability p and
  - we get A_- with the remaining probability $1-p$.

2. Traditional Decision Theory (cont-d)

- For $p = 0$, the lottery $L(p)$ coincides with the very bad alternative A_-.
- Thus, $L(0)$ is worse that any actual alternative A; we will denote this by $L(0) = A_- < A$.
- For $p = 1$, the lottery $L(p)$ coincides with the very good alternative A_+.
- Thus, $L(1)$ is better than any actual alternative A: $A < A_+$.
- We can ask the user to compare the alternative A with the lotteries $L(p)$ corresponding to different p.

3. Traditional Decision Theory (cont-d)

- We assume that for every two alternatives A and B, the user always decides:
  - whether A is better (i.e., $B < A$)
  - or whether B is better (i.e., $A < B$),
  - or whether A and B are of the same quality to this user; we will denote this by $A \sim B$.
- We also assume that the user's decisions are consistent:
  - that the preference relation $<$ is transitive and
  - that $p < p'$ implies $L(p) < L(p')$.
- One can see that under these assumptions, there is a threshold value p_0 such that:
  - for all $p < p_0$, we have $L(p) < A$, and
  - for all $p < p_0$, we have $A < L(p)$.

4. Traditional Decision Theory (cont-d)

- This threshold value is called the *utility* of the alternative A. The utility is usually denoted by $u(A)$.
- By definition of the utility, for every small value $\varepsilon > 0$, we have $L(u(A) - \varepsilon) < A < L(u(A) + \varepsilon)$.
- For very small ε, lotteries with probabilities $u(A)$ and $u(A) \pm \varepsilon$ are practically indistinguishable.
- So we can say that the alternative A is equivalent to the lottery $L(u(A))$.
- We will denote this equivalence by $A \equiv L(u(A))$.
- Suppose now that for some action a, we have consequences $A_1, \ldots, A_n$ with probabilities $p_1, \ldots, p_n$.
- This means that the action a is equivalent to a lottery in which we get each A_i with probability p_i.

5. Traditional Decision Theory (cont-d)

- Each alternative A_i, in its turn, is equivalent to the lottery $L(u(A_i))$ in which:
  - we get A_+ with probability $u(A_i)$ and
  - we get A_- with the remaining probability $1-u(A_i)$.
- Thus, the action a is equivalent to a two-stage lottery, in which:
  - first, we select A_i with probability p_i, and
  - then, depending on A_i, we select A_+ with probability $u(A_i)$ and A_- with probability $1-u(A_i)$.
- As a result of this two-stage lottery, we get either A_+ or A_-, and the probability of getting A_+ is equal to

$$p=\sum_{i=1}^{n} p_i \cdot u(A_i).$$

6. Traditional Decision Theory (cont-d)

- So, the action a is equivalent to the lottery $L(p)$.
- By definition of utility, this means that the utility $u(a)$ of the action a is equal to this probability p, i.e., that

$$u(a) = \sum_{i=1}^{n} p_i \cdot u(A_i).$$

- By definition of utility, we select the action with the largest possible value of utility.
- The right-hand side of the above formula is the expected value of the utility.
- So, a rational person should select the alternative with the largest possible value of expected utility.

7. Utility Is Defined Modulo a Linear Transformation

- Numerical values of utility depend on the selection of the very bad and very good alternatives A_- and A_+.
- What if we select a different pair A'_- and A'_+?
- Let us consider the case $A_- < A'_- < A'_+ < A_+$.
- Every alternative A is equivalent to a lottery $L'(u'(A))$ in which:
  - we get A'_+ with probability $u'(A_+)$ and
  - we get A'_- with probability $1 - u'(A)$.
- A'_- is equivalent to a lottery $u(A'_-)$ in which:
  - we get A_+ with probability $u(A'_-)$ and
  - we get A_- with the remaining probability $1-u(A'_-)$.

8. Linear Transformation (cont-d)

- A'_+ is equivalent to a lottery $u(A'_+)$ in which:
  - we get A_+ with probability $u(A'_+)$ and
  - we get A_- with the remaining probability $1-u(A'_+)$.
- Thus, the original alternative A is equivalent to a two-stage lottery in which:
  - we first select A'_- or A'_+, and
  - then, depending on what we selected on the first stage, select A_+ or A_-.
- As a result of this two-stage lottery, we get either A_+ or A_-; the probability p of selecting A_+ is equal to

$$p = u'(A) \cdot u(A'_+) + (1 - u'(A)) \cdot u(A'_-) =$$
$$u(A'_-) + u'(A) \cdot (u(A_+) - u(A_-)).$$

- By definition of utility, this probability p is the utility $u(A)$ of the alternative A in terms of A_- and A_+.

9. Linear Transformation (cont-d)

- Thus, $u(A) = u(A'_-) + u'(A) \cdot (u(A_+) - u(A_-))$.
- In other words, the utility is defined modulo a linear transformation.
- This is similar to measuring quantities like time or temperature, where the numerical value depends:
  - on the selection of the starting point and
  - on the selection of the measuring unit.

10. What If We Only Have Partial Information About Probabilities: Two Approaches

- The formula for the expected utility assumes that we know the probability of each alternative.
- In many practical situations, we only have partial information about the probabilities.
- In this case, for different probability distributions $P = (p_1, \ldots, p_n)$, we have different expected utility.
- If two probability distributions P and P' are possible, then we can also consider as possible the case when:
  - we have P with probability β and
  - we have P' with probability $1 - \beta$.
- In this case, the expected utility is equal to the convex combination of expected utilities corr. to P and P'.

11. Two Approaches (cont-d)

- So, the set of possible values of expected utility is closed under convex combinations.
- It is, thus, an interval $[\underline{u}(a), \overline{u}(a)]$.
- How can we make a decision under this interval uncertainty?
- There are two approaches to solving this problem.
- The 1st approach is based on the fact that we need to compare the action a, in particular, with lotteries $L(p)$.
- Thus, we need to assign, to this interval, a corresponding utility $u(a)$:

$$u(a) = f(\underline{u}(a), \overline{u}(a)) \text{ for some functions } f(x, y).$$

- Utility is defined modulo a linear transformation

$$u(a) \to k \cdot u(a) + \ell, \text{ for some } k > 0 \text{ and } \ell.$$

12. Two Approaches (cont-d)

- It is therefore reasonable to require that the function $f(x, y)$ not depend on the selection of A_- and A_+:
  - if $z = f(x, y)$,
  - then $z' = f(x', y')$ for $z' = k \cdot a + \ell$, $x' = k \cdot x + \ell$, and $y' = k \cdot y + \ell$.
- In particular, every interval $[\underline{a}, \overline{a}]$ can be obtained from the interval $[0, 1]$ if we take $k = \overline{u} - \underline{u}$ and $\ell = \underline{u}$.
- So, for $\alpha \stackrel{\text{def}}{=} f(0, 1)$, the above invariance implies:

$$f(\underline{u}, \overline{u}) = \alpha \cdot (\overline{u} - \underline{u}) + \underline{u} = \alpha \cdot \overline{u} + (1 - \alpha) \cdot \underline{u}.$$

- Thus, we should select the action a for which the value $\alpha \cdot \overline{u} + (1 - \alpha) \cdot \underline{u}$ is the largest.
- This idea was first proposed by a Nobelist Leo Hurwicz.
- It is known as *Hurwicz optimism-pessimism approach.*

Traditional Decision . . .
Utility Is Defined . . .
What If We Only Have . . .
Two Envelopes Problem
How Realistic Is This . . .
At First Glance, This . . .
This Is Not Really a . . .
Idealized Formulation: . . .
So What Should We Do?
Home Page
Title Page
◀◀ ▶▶
◀ ▶
Page 13 of 36
Go Back
Full Screen
Close
Quit

13. Two Approaches (cont-d)

- When $\alpha = 1$, the decision maker only takes into account the best possible situation, with utility $\overline{u}$.
- This is what we usually mean by extreme optimism.
- When $\alpha = 0$, the decision maker only takes into account the worst possible situation, with utility $\underline{u}$.
- This is what we usually mean by extreme pessimism.
- In real life, these two behaviors do not make sense:
  - extreme pessimism means not going into the street at all – a car may hit;
  - extreme optimism would mean crossing the street on red light in heavy traffic.
- Realistic decision making corresponds to values α between 0 and 1.
- For example, it is often recommended to select $\alpha = 0.5$.

14. Two Approaches (cont-d)

- An alternative approach is that:
  - instead of considering all possible probability distributions,
  - we should select the most reasonable one.
- Some of the possible distributions correspond to larger uncertainty, some to smaller uncertainty.
- We do not want to pretend that we have less uncertainty.
- So, it is reasonable to select the distribution with the largest possible value of uncertainty.
- A natural measure of uncertainty of a probability distribution is its entropy.
- It is the average number of binary questions that we need to ask to determine the actual value.

15. Two Approaches (cont-d)

- Thus, out of all possible distributions, we select the one with the largest possible entropy.
- This approach is known as the *Maximum Entropy* approach.
- In particular:
  - when we have a natural symmetry,
  - the resulting distribution is invariant with respect to the same symmetry.

16. Two Approaches (cont-d)

- For example:
  - if we class of distributions is invariant with respect to permutations,
  - the maximum entropy distribution is also invariant with respect to all permutations,
  - and thus, assigns equal probability to all alternatives.
- This is known as Laplace Indeterminacy Principle.
- This invariance idea can be applied to the case when:
  - all we know is that the value x of some physical quantity is positive,
  - but we have no information about different probabilities.

17. Two Approaches (cont-d)

- Which probability density function (pdf) $\rho(x)$ should we choose?
- The numerical value of x depends on the choice of a measuring unit.
- If we select a unit which is λ times smaller, then all numerical values x are replaced by new values $x' = \lambda \cdot x$.
- For example, 1.7 meters becomes $100 \cdot 1.7 = 170$ cm.
- Under this transformation, the original pdf $\rho(x)$ takes, in the new unit, the form $\frac{1}{\lambda} \cdot \rho\left(\frac{x'}{\lambda}\right)$.
- We want to have the pdf that does not depend on the choice of the measuring unit.

18. Two Approaches (cont-d)

- So we should have $\frac{1}{\lambda} \cdot \rho\left(\frac{x'}{\lambda}\right) = \rho(x')$, or, equivalently,
$$\rho\left(\frac{x'}{\lambda}\right) = \lambda \cdot \rho(x').$$
- For $x' = 1$ and $\lambda = 1/a$, we get $\rho(a) = \text{const} \cdot \frac{1}{a}$.
- Strictly speaking, this is *not* a probability density function, since here, we have $\int \rho(a)\,da = \infty \neq 1$.
- This is known as an *improper* probability distribution.
- Both Hurwicz and Maximum Entropy approaches have been used in many practical problems.
- They usually lead to intuitively acceptable results, even when we ignore some available information.
- E.g., when this information is too vague to be easily formalized.

19. Two Approaches (cont-d)

- In this talk, we show, on the example of the two envelopes problems, that:
  - if we ignore this information,
  - both approaches lead to a counter-intuitive results.
- So, when making a decision under uncertainty, we must take into account all available information.

20. Two Envelopes Problem

- Someone places some amount of money in one envelope, and a double that amount in another envelope.
- We do not know which envelope contains a smaller amount and which contains a larger amount.
- We can pick one envelope and check how much money x it has.
- Then, we can make a decision:
  - we can either keep this amount
  - or we can select the second envelope instead.
- The question is:
  - should we keep the original amount or
  - should we select the other envelope instead?

21. How Realistic Is This Situation

- The above situation is over-simplified, but similar situations occur in real life.
- For example, suppose that there are two competing countries producing certain military equipment.
- A smaller country would like to buy from one of them.
- For political reasons, it cannot negotiate simultaneously with both.
- So it starts serious negotiations with one of the countries.
- After negotiations, it comes up with the expected cost x of the purchase.
- This cost is usually rather significant.

22. How Realistic Is This Situation (cont-d)

- Now, this smaller country has a choice:
  - it can go with this contract, or
  - it can try its luck by negotiating with the competing country.
- If these negotiations do not lead to cheaper prices, there is no going back to the original contract.
- What should it do?
- More peaceful examples can be found; e.g.:
  - when forming political alliances inside a country or within countries,
  - when planning mergers, etc.

23. At First Glance, This Is Somewhat Paradoxical

- Let us go back to the original two envelopes problem.
- From the common sense viewpoint, since our selection was random, it does not make sense to switch.
- On the other hand, intuitively, we do not know these the second envelope has $2x$ or $x/2$.
- In line with the Laplace Indeterminacy Principle, we assume that these two amounts have equal prob. 0.5.
- The expected gain is $0.5 \cdot 2x + 0.5 \cdot (x/2) = 1.25x$.
- Since $1.25x > x$, it looks like it is always reasonable to switch – which contradicts to common sense.

24. Paradox (cont-d)

- This argument is based on the assumption that utility is proportional to money.
- In practice, it is proportional to square root of money.
- However, the paradox remains if we take $\sqrt{2} \cdot x$ and $x/\sqrt{2}$ as the utilities of the two alternatives.

25. This Is Not Really a Paradox

- A detailed analysis shows that, in reality, there is no paradox.
- Indeed, suppose that the original values come with the probability density $\rho(x)$.
- Then, the double amounts come with probability density $\dfrac{1}{2} \cdot \rho\left(\dfrac{x}{2}\right)$.
- Suppose that we found the value x.
- Then, the condition probability that this is the original amount of money is proportional to $\rho(x)$.
- Here, the remaining envelope contains $2x$.
- The conditional probability that this is the double amount of money is proportional to $\dfrac{1}{2} \cdot \rho\left(\dfrac{x}{2}\right)$.
- Then, the remaining envelope contains $x/2$.

26. This Is Not Really a Paradox (cont-d)

- Since these two probabilities should add up to 1, we conclude that they should be equal to

$$\frac{\rho(x)}{\rho(x)+\frac{1}{2}\cdot\rho\left(\frac{x}{2}\right)} \text{ and } \frac{\frac{1}{2}\cdot\rho\left(\frac{x}{2}\right)}{\rho(x)+\frac{1}{2}\cdot\rho\left(\frac{x}{2}\right)}.$$

- Thus, the expected value of the gain that we get when we switch is equal to

$$\frac{\rho(x)}{\rho(x)+\frac{1}{2}\cdot\rho\left(\frac{x}{2}\right)}\cdot 2x+\frac{\frac{1}{2}\cdot\rho\left(\frac{x}{2}\right)}{\rho(x)+\frac{1}{2}\cdot\rho\left(\frac{x}{2}\right)}\cdot\frac{x}{2}.$$

- If this amount is smaller that x, we should not switch.
- If this amount is larger than x, we should switch.
- If this amount is equal to x, it does not matter whether we switch or not.

27. This Is Not Really a Paradox (cont-d)

- If we know the probability distribution, then, depending of the amount x, we should switch or not switch.
- E.g., if we know that all the values x are smaller than some amount x_0, then:
  - if we get an amount $x > x_0$, we know that this is the larger amount,
  - so switching does not make sense.
- Similarly, if we know that all original amounts are larger than or equal to some minimal amount m:
  - we have a value $x < 2m$, we know that this cannot be the double amount,
  - so we should switch.

28. This Is Not Really a Paradox (cont-d)

- The above formula explains why there is no paradox.
- The only time when the probability of each option is exactly 1/2 is when $\rho(x) = \frac{1}{2} \cdot \rho\left(\frac{x}{2}\right)$ for all x.
- Since this can be repeated not just for doubling money, but also for λ times larger amount, we should have:

$$\rho(x) = \frac{1}{\lambda} \cdot \rho\left(\frac{x}{\lambda}\right) \text{ for all } x \text{ and } \lambda.$$

- We already know that this leads to an "improper" – not real – probability distribution.

29. Idealized Formulation: What the Two Approaches Recommend

- Suppose now that we do not have any information about the probability distributions.
- What should the above two approaches recommend?
- Let us first consider the Hurwicz approach.
- We do not know the pdf, so we should consider all possible probability density functions.
- The worst-possible case is when $\frac{1}{2} \cdot \rho\left(\frac{x}{2}\right) = 0$, then after switching, we get $x/2$ with probability 1.
- The best-possible case is when $\rho(x) = 0$, then after switching, we get $2x$ with probability 1.
- So, e.g., for $\alpha = 1/2$, the Hurwicz combination is equal to $0.5 \cdot 2x + 0.5 \cdot (x/2) = 1.25x > x$.

30. Idealized Formulation (cont-d)

- So we arrive at the counter-intuitive conclusion that we should always switch.
- So maybe we should use smaller value of Hurwicz's α?
- This will not help, since, as we have mentioned, we could have $\lambda \cdot x$ in the second envelope.
- In this case, the expected value after switching is $\alpha \cdot \lambda \cdot x + (1 - \alpha) \cdot (x/\lambda)$.
- For any $\alpha > 0$, for a sufficiently large λ, we get a counter-intuitive conclusion that we should switch.
- The only case when this conclusion is not possible is $\alpha = 0$ which is not realistic at all.
- So, for this problem, Hurwicz approach leads to a counter-intuitive behavior.

31. Idealized Formulation (cont-d)

- Let us now consider the maximum entropy approach.
- Since all we know is that the amount is positive, we should use the corr. improper distribution; then

$$\rho(x) = \frac{1}{2} \cdot \rho\left(\frac{x}{2}\right).$$

- So we also get $1.25x$.
- In other words, in this case, the Maximum Entropy approach also leads to a counter-intuitive conclusion.

32. So What Should We Do?

- Why did we get the counter-intuitive result?
- Because our description of the problem is not realistic.
- Do we really believe that an envelope can contain any amount of money?
- Realistically, an envelope cannot contain more than several thousand dollars.
- A million will not fit into an envelope.
- So, we can impose an upper limit x_0 on the original amount of money.
- Then the conclusions change – and become more intuitive.

33. What Should We Do (cont-d)

- For the Hurwicz approach with $\alpha = 0.5$, we still recommend switching when $x \le x_0$.
- However, now we do not recommend switching when $x > x_0$, since in this case, $\rho(x) > 0$.
- The Maximum Entropy approach leads to the uniform distribution

$$\rho(x) = \text{const for } x \le x_0 \text{ and } \rho(x) = 0 \text{ for } x > x_0.$$

- For this distribution, the above formula also recommends switching if and only if $x \le x_0$.
- This common recommendation seems to be in perfect accordance with common sense.

34. Conclusion

- When making decision under uncertainty:
  - it is important to take into account *all* available information,
  - even seemingly useless one.
- Otherwise, if we ignore this information, we may end up with counter-intuitive recommendations.

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