Computing with Words: Towards a New Tuple-Based Formalization

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Need to Use Words
Need for Data Processing
Need for Computing . . .
From a Words-Related . . .
Resulting Fuzzy-Based . . .
A Seemingly Natural . . .
Main Idea
Main Proposition: For . . .
Examples

1. Need to Use Words

- Often, to describe height etc., we use words such as "small", "medium", "high", etc.
- If we only use the selected words $w_1, \ldots, w_n$, we get a rather crude description of the quantity.
- A more accurate description may include several words, with degrees associated with different words.
- Example: rather short, but closer to medium height.
- We can describe this by specifying degrees d_i to which the quantity fits each word w_i.
- Then, our opinion of each value is described by a tuple of degrees $d = (d_1, \ldots, d_n)$.

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2. Need for Data Processing

- Often, we are interested in the value of a physical quantity y which is difficult to measure directly.
- For example, we are interested in tomorrow's temperature.
- In such situations, a usual approach is:
  - find easier-to-estimate quantities $x_1, \ldots, x_m$ related to y by a known dependence $y = f(x_1, \ldots, x_m)$, and
  - to use the estimates of x_i to compute the estimate for y.
- This computation of y based on $x_1, \ldots, x_m$ is known as *data processing*.

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3. Need for Computing With Words

- When the estimates for x_j are given in the form of tuples d, we face the following problem:
  - we know the tuples $d^{(j)} = \left(d_1^{(j)}, \ldots, d_n^{(j)}\right)$ which describes our knowledge about each input x_j;
  - we want to describe the resulting knowledge about y in a similar tuple form.
- In particular:
  - we have quantities x_1 and x_2 characterized by the tuples $d^{(1)}$ and $d^{(1)}$;
  - we want to compute tuples corresponding to $x_1 + x_2$, $x_1 - x_2$, $x_1 \cdot x_2$, etc.
- In general, instead of computing with numbers, we should be able to compute with words (L. Zadeh).

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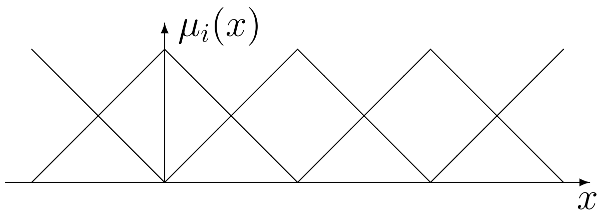
4. How to Represent the Original Words

- A natural way to represent the original words in the computer-understandable form is to use fuzzy logic.
- Usually, the corresponding membership functions $\mu_i(x)$ are triangular, and different functions differ by a shift.
- In precise terms, for some some starting point s and step h, we have

$$\mu_i(x) = \max\left(0, 1 - \frac{|x - (s + i \cdot h)|}{h}\right):$$



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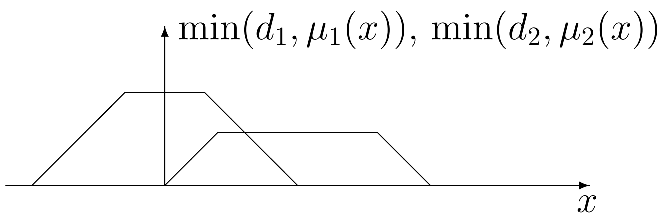
5. From a Words-Related Tuple Representation to a Membership Function

- A tuple $d = (d_1, \ldots, d_n)$ represents a value x if *one of* the following conditions hold:
  - the quantity q is characterized by the word w_1, *and* x satisfies the property described by this word, ...
  - the quantity q is characterized by the word w_n, *and* x satisfies the property described by this word.
- If we use min for "and" and max for "or", we get

$$\mu_d(x) = \max(\min(d_1, \mu_1(x)), \ldots, \min(d_n, \mu_n(x))).$$



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6. Resulting Fuzzy-Based Formalization of Computing with Words

- We know the tuples $d^{(1)}$ and $d^{(2)}$ describing the two quantities x_1 and x_2; then:
  - first, we generate membership functions $\mu^{(1)}(x_1)$, $\mu^{(2)}(x_2)$ corresponding to the tuples $d^{(1)}$, $d^{(2)}$;
  - then, we use Zadeh's extension principle to compute the membership f-n $\mu(x)$ corr. to $y = f(x_1, x_2)$;
  - finally, we generate the tuple d corresponding to the resulting membership function $\mu(x)$.
- To implement this idea, we need to generate a tuple corresponding to a given membership function.

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7. A Seemingly Natural Idea and Its Limitations

- We look for the degree d_i to which it's possible that:
  - a quantity described by a membership function $\mu(x)$
  - is in agreement with w_i.
- This means that *some* value x is in agreement with the membership function *and* with the word w_i.
- If we use min for "and" and max for "or", we get

$$d'_i = \max_x(\min(\mu(x), \mu_i(x))).$$

- Example: we start with the word w_i, i.e., with the tuple $d = (0, \ldots, 0, 1, 0, \ldots, 0)$.
- Then, for $f(x) = x$, we would like to get d back.

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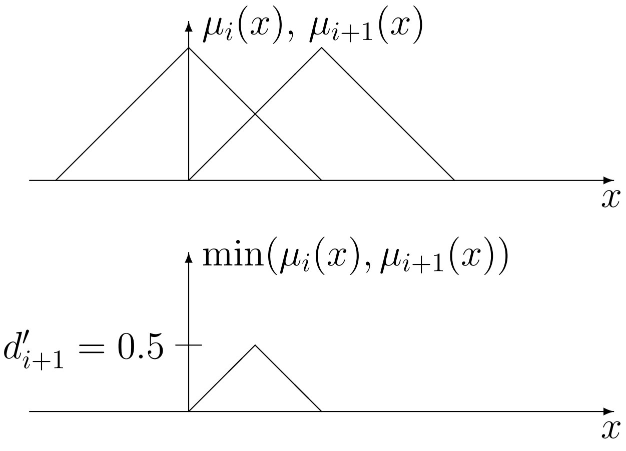
8. A Seemingly Natural Idea and Its Limitations (cont-d)

- We start with the word w_i: $d = (0, \ldots, 0, 1, 0, \ldots, 0)$.
- We compute $d'_i = \max\limits_x(\min(\mu(x), \mu_i(x)))$, and get

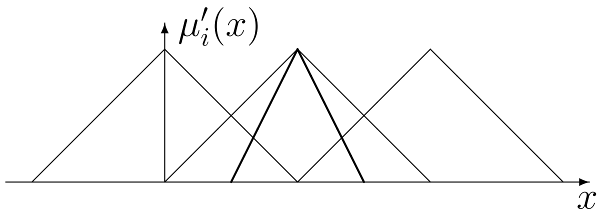
$$d'_i = (0, \ldots, 0, 0.5, 1, 0.5, 0 \ldots, 0) \neq d :$$



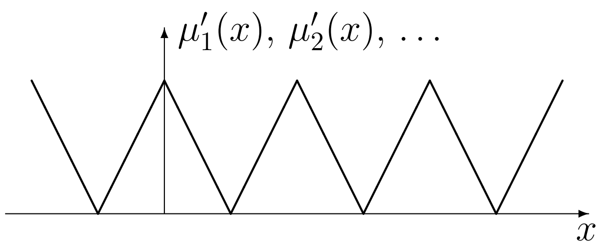
9. Main Idea

- *Problem:* membership f-s $\mu_i(x)$ and $\mu_{i+1}(x)$ intersect.
- *Solution:* remove the intersecting parts, i.e., take

$$\mu'_i(x) = \max(0, \mu_i(x) - \max(\mu_{i-1}(x), \mu_{i+1}(x))) :$$



- The reduced functions $\mu'_i(x)$ no longer overlap:



10. Main Idea (cont-d)

- Instead of the original functions $\mu_i(x)$, we compute the reduced functions

$$\mu_i'(x) = \max(0, \mu_i(x) - \max(\mu_{i-1}(x), \mu_{i+1}(x))).$$

- Similarly, instead of the membership function $\mu(x)$, we compute the reduced function

$$\mu'(x) = \max(0, \mu(x) - \max(\mu_{i-1}(x), \mu_{i+1}(x))).$$

- Then, we compute the degrees based on these reduced functions, as

$$d_i' = \max_x(\min(\mu'(x), \mu_i'(x))).$$

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11. Main Proposition: For $f(x) = x$, We Get the Tuple d Back

- Let $\mu_i(x)$ be a sequence of triangular functions.
- Let $d = (d_1, \ldots, d_n)$ be a tuple of numbers $d_i \in [0, 1]$.
- Let $\mu_d(x)$ be the corresponding membership function

$$\mu_d(x) = \max(\min(d_1, \mu_1(x)), \ldots, \min(d_n, \mu_n(x))).$$

- We then:
  - compute $\mu'_i(x) \stackrel{\text{def}}{=} \max(0, \mu_i(x) - \max(\mu_{i-1}(x), \mu_{i+1}(x)))$;
  - compute the reduced functions

$$\mu'(x) = \max(0, \mu_d(x) - \max(\mu_{i-1}(x), \mu_{i+1}(x))),$$

  - and apply the formulas $d'_i = \max\limits_x(\min(\mu'(x), \mu'_i(x)))$.
- As a result, we get $d'_i = d_i$ for all i.

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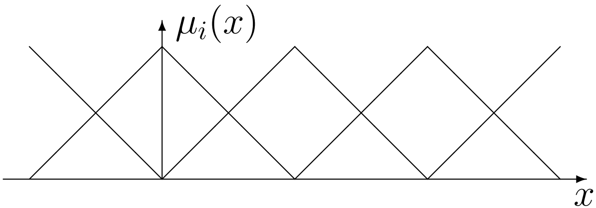
12. Beyond Triangular Membership Functions

- We formulated our Main Proposition for *triangular* membership functions.



- A similar result holds for *any* set of membership functions $\mu_i(x)$ for which, for some sequence of values t_i:
  - $\mu_i(t_i) = 1$, and
  - $\mu_i(x)$ is only different from 0 for $x \in [t_{i-1}, t_{i+1}]$.

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13. Resulting Definition of an Operation with Tuples

- We *know:*
  - tuples $d^{(i)}$ describing different quantities x_i;
  - an algorithm $y = f(x_1, \ldots, x_n)$.
- We *compute* a tuple d corresponding to $y = f(x_1, \ldots, x_n)$ as follows:
  - first, we compute the membership functions $\mu_i(x_i)$ corresponding to the tuples $d^{(i)}$;
  - we apply Zadeh's extension principle to $\mu_i(x_i)$ to compute the membership function $\mu(y)$ for

$$y = f(x_1, \ldots, x_n);$$

  - we then apply the reduced-functions formula to $\mu(y)$ and get the desired tuple d.

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14. Examples

1. We *add* two words $w_{i'}$ and $w_{i''}$:

   - here, $d^{(1)} = (0, \ldots, 0, 1, 0, \ldots, 0)$ (1 on i-th place) and $d^{(2)} = (0, \ldots, 0, 1, 0, \ldots, 0)$ (1 on i'-th place);
   - we get d s.t. $d_{i'+i''} = 1$, $d_{(i'+i'')-1} = d_{(i'+i'')+1} = 0.5$, and $d_j = 0$ for all other j:

   $$d = (0, \ldots, 0, 0.5, 1, 0.5, 0, \ldots, 0).$$

2. We *subtract* $w_{i'}$ and $w_{i''}$; we get a tuple with $d_{i'-i''} = 1$, $d_{(i'-i'')-1} = d_{(i'-i'')+1} = 0.5$, and $d_j = 0$ for other j:

   $$d = (0, \ldots, 0, 0.5, 1, 0.5, 0, \ldots, 0).$$

3. A *shift* $f(x) = x + a \cdot h$ $(0 < a < 1)$ of a word w_i leads to the tuple $d = (0, \ldots, 0, 1 - a, a, 0, \ldots, 0)$.

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15. Acknowledgment

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