A Symmetry-Based Approach to Selecting Membership Functions and Its Relation to Chemical Kinetics

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1. Shift-Invariant Quantities: A Brief Reminder

- In many physical theories, there is no fixed starting point for measuring the corr. physical quantities.
- We can measure time based on the current calendar or starting with 1789 (the year of the French revolution).
- If we select a new one which is q_0 units smaller, then the original numerical value q changes into $q' = q + q_0$.
- For such quantities, all the properties do not change if we change this starting point, i.e., if replace q by $q + q_0$.
- Strictly speaking, there is the absolute starting point for measuring time: the Big Bang.
- However, in most cases, the equations remains the same if we change a starting point for time.
- Similarly, in many practical applications, there is no absolute starting point for measuring potential energy.

2. Ideally, Membership Functions Should Reflect This Symmetry

- Often, our knowledge is imprecise (“fuzzy”).
- To describe and process such knowledge, L. Zadeh invented the ideas of fuzzy sets.
- A fuzzy set on a universal set X is characterized by its membership function $\mu : X \to [0, 1]$.
- For shift-invariant quantities, our selection of $\mu(x)$ should reflect shift-invariance.
- A seemingly natural idea is to require that $\mu(x)$ be shift-invariant, i.e., $\mu(q) = \mu(q + q_0)$ for all q and q_0.
- Unfortunately, the only membership function $\mu(q)$ which satisfies this condition is the constant function.
- So, we cannot require that a *single* membership function is shift-invariant.

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3. A Collection of Several Membership Functions Should Be Shift-Invariant

- A *single* membership function cannot be shift-invariant.
- It is thus reasonable to require that a collection of *several* membership functions is shift-invariant.
- In T-S fuzzy control, we often end up with a linear combination $\sum c_i \cdot \mu_i(q)$ of these membership functions.
- Thus, we consider sets of all such linear combinations – a *linear space*.
- We are therefore looking for shift-invariant linear spaces.

4. Definitions

- By a *finite-dimensional linear space*, we mean the class of all functions of the type $\sum\limits_{i=1}^{n} c_i \cdot \mu_i(q)$, where:
  - $n \geq 1$,
  - differentiable functions $\mu_1(q)$, …, $\mu_n(x)$ are fixed (and assumed to be linearly independent), and
  - the coefficients $c_1, \ldots, c_n$ can take any real values.
- We say that a linear space L is *shift-invariant* if
  - for every function $f(q)$ from the space L and
  - for every real number q_0,
  - the function $f(q+q_0)$ also belongs to the class L.
- We say that a shift-invariant linear space L is *basic* if $L \neq \text{Lin}(L_1 \cup L_2)$ for shift-inv. linear spaces L_1, L_2.

5. Main Result

- We say that a linear space of functions L is *fuzzy-related* if the following two conditions hold:
    - L is the set of all linear combinations of functions $\mu_1(q), \ldots, \mu_n(q)$ s.t. $\mu_i(q) \in [0,1]$ for all $q \geq 0$.
    - L does not include the constant functions $f(q) \equiv 1$.
- **Proposition.** *Each basic shift-invariant fuzzy-related linear space L is a linear combination of functions*

$$\mu_1(q) = \exp(-\lambda \cdot q), \mu_2(q) = q \cdot \exp(-\lambda \cdot q), \ldots,$$

$$\mu_i(q) = q^{i-1} \cdot \exp(-\lambda \cdot q), \ldots, \mu_n(q) = q^{n-1} \cdot \exp(-\lambda \cdot q),$$

for some $\lambda > 0$.

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6. Proof of the Main Result: Ideas

- Shift-invariance means that for every q_0, there are some $c_{ij}(q_0)$ for which $\mu_i(q+q_0)=\sum\limits_{j=1}^{n} c_{ij}(q_0)\cdot\mu_j(q)$.
- Differentiating both sides w.r.t. q_0 and taking $q_0=0$, we get a system of linear differential equations
$$\mu_i'(q)=\sum_{j=1}^{n} C_{ij}\cdot\mu_j(q).$$
- Solutions to such systems are well-known: they are linear combinations of expressions $x^k\cdot\exp(-\lambda\cdot q)$.
- Expressions corresponding to different λ form shift-invariant spaces.
- Thus, since L is basic, we can only have one value λ.
- The restrictions that $\mu_i(q)\in[0,1]$ for all $q\ge 0$ and $\mu(q)\not\equiv 1$ imply that $\lambda>0$. Q.E.D.

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7. What Is Chemical Kinetics: Brief Reminder

- Chemical kinetics describes the change in concentration of chemical substances.
- The reaction rate is proportional to the product of concentrations of reagents.
- For example, for a reaction $A + B \to C$, the reaction rate is proportional to the product $a \cdot b$.
- Due to this reaction rate $k \cdot a \cdot b$:
  - the amounts a and b of substances A and B decrease with this rate, while
  - the amount c of the substance C increases with this rate:
    $$\frac{da}{dt} = -k \cdot a \cdot b; \quad \frac{db}{dt} = -k \cdot a \cdot b; \quad \frac{dc}{dt} = k \cdot a \cdot b.$$
- If we have several reactions, then we add the rates corresponding to different reactions.

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8. Relation Between Membership Functions and Chemical Kinetics: An Intuitive Idea

- Let us consider "small", "medium", and "large".
- The value $q = 0$ is absolutely small. As we increase q:
    - what was originally small starts slowly transforming into medium: $s \to m$;
    - then, what was originally medium starts slowly transforming into large: $m \to \ell$, etc.
- It is reasonable to assume that both "chemical reactions" $s \to m$ and $m \to \ell$ have the same rate k, then:

$$\frac{ds}{dq} = -k \cdot s; \quad \frac{dm}{dq} = k \cdot s - k \cdot m; \quad \frac{d\ell}{dq} = k \cdot m.$$

- It is natural to interpret the "concentrations" $s(q)$, $m(q)$, …, as degrees to which q is small, medium, …
- In other words, we take $\mu_{\text{small}}(q) = s(q)$, $\mu_{\text{medium}}(q) = s(q)$, …

9. Connection Between Chemical Kinetics and Membership Functions: General Case

- Let's consider $n \geq 3$ membership functions $\mu_1(q)$, …, $\mu_n(q)$, with reactions $\mu_1 \to \mu_2, \ldots, \mu_{n-1} \to \mu_n$.
- The corresponding equations of chemical kinetics have the form:

$$\frac{d\mu_1(q)}{dq} = -\lambda\cdot\mu_1(q), \ldots, \frac{d\mu_i(q)}{dq} = \lambda\cdot\mu_{i-1}(q) - \lambda\cdot\mu_i(q), \ldots,$$

$$\frac{d\mu_n(q)}{dq} = \lambda\cdot\mu_{n-1}(q).$$

- The initial values are $\mu_1(0) = 1$ and $\mu_2(0) = \ldots = \mu_n(0) = 0$.
- Thus, this system allows us to uniquely determine the values $\mu_i(q)$ for all $q \geq 0$, as

$$\mu_i(q) = \frac{\lambda^{i-1}}{(i-1)!}\cdot q^{i-1}\cdot\exp(-\lambda\cdot q).$$

10. Comment about Normalization

- The membership f-s $\mu_i(q) = \dfrac{\lambda^{i-1}}{(i-1)!} \cdot q^{i-1} \cdot \exp(-\lambda \cdot q)$ are not normalized: $\max\limits_q \mu_i(1) \neq 1$.
- We can normalize each function by dividing it by its maximum value.
- The equation $\dfrac{d\mu_i(q)}{dq} = 0$ leads to $q_{\max} = (i-1) \cdot \dfrac{1}{\lambda}$, hence $\mu_i(q_{\max}) = \dfrac{(i-1)^{i-1}}{(i-1)!} \cdot \exp(-(i-1))$.
- So, the normalized membership functions have the form

$$\widetilde{\mu}_i(q) = \frac{\lambda^{i-1}}{(i-1)^{i-1}} \cdot q^{i-1} \cdot \exp((i-1) - \lambda \cdot q).$$

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11. Discussion

- Two different approaches lead to the same class of membership functions $\mu_i(q) = \text{const} \cdot q^{i-1} \cdot \exp(-\lambda \cdot q)$:
  - the approach based on shift-invariance, and
  - an analogy between chemical kinetics and fuzzy logic.
- The fact that two different ideas lead to the same membership f-s $\mu_i(q)$ confirms that these f-s are reasonable.
- It is worth mentioning that these functions $\mu_i(q)$ have equally spaced maxima $q_{\max} = (i-1) \cdot \dfrac{1}{\lambda}$.
- This is good news, since people normally use membership functions $\mu_i(q)$ with equally spaced maxima.

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