r-Bounded Fuzzy Measures are Equivalent to ε-Possibility Measures

Karen Richart, Olga Kosheleva, and Vladik Kreinovich

University of Texas at El Paso
El Paso, TX 79968, USA
karichart@miners.utep.edu, olgak@utep.edu,
vladik@utep.edu

1. Outline

- Traditional probabilistic description of uncertainty is based on *additive* probability measures.
- To describe non-probabilistic uncertainty, it is therefore reasonable to consider *non-additive* measures.
- An important class of non-additive measures are *possibility* measures, for which $\mu(A \cup B) = \max(\mu(A), \mu(B))$.
- In this talk, we show that possibility measures are, in some sense, universal approximators:
  - for every $\varepsilon > 0$,
  - every non-additive measure which satisfies a certain reasonable boundedness property
  - is equivalent to a measure which is ε-close to a possibility measure.

2. Additive Measures: Brief Reminder

- Let X be a set called a *universal set*.
- An *algebra* $\mathcal{A}$ is a non-empty class of subsets $A \subseteq X$ which is closed under complement, $\cup$, and $\cap$, i.e.:
  - if $A \in \mathcal{A}$, then its complement $-A$ is also in $\mathcal{A}$;
  - if $A \in \mathcal{A}$ and $B \in \mathcal{A}$, then $A \cup B \in \mathcal{A}$;
  - if $A \in \mathcal{A}$ and $B \in \mathcal{A}$, then $A \cap B \in \mathcal{A}$.
- An *additive measure*, we mean a function μ mapping sets $A \in \mathcal{A}$ to numbers $\mu(A) \geq 0$ s.t. $A \cap B = \emptyset$ implies

$$\mu(A \cup B) = \mu(A) + \mu(B).$$

- *Examples:* length, area, volume, probability.
- *Properties:* monotonicity ($A \subseteq B$ implies $\mu(A) \leq \mu(B)$), $\mu(\emptyset) = 0$.

3. Fuzzy Measures: Brief Reminder

- A function $\mu(A)$ defined on an algebra of subsets of a universal set X is called a *fuzzy measure* if:
  - it is monotonic, i.e., $A \subseteq B$ implies $\mu(A) \le \mu(B)$, and
  - it satisfies the properties $\mu(\emptyset) = 0$ and $\mu(X) = 1$.
- A function $\mu(A)$ is called *maxitive* if for every two sets A and B, we have $\mu(A \cup B) = \max(\mu(A), \mu(B))$.
- By a *possibility measure*, we mean a maxitive fuzzy measure.

4. Which Measures Are Reasonable?

- In general, a measure $\mu(A)$ describes how important is the set A.
- The larger the measure, the more important is the set A.
- From this viewpoint:
  - if we take the union $A \cup B$ of two sets of bounded size,
  - then the size of the union cannot be arbitrarily large.
- The size of $A \cup B$ should be limited by some bound depending on the bound on $\mu(A)$ and $\mu(B)$.
- Similarly, if the sizes of A and B are sufficiently small, then the size of the union should also be small.

5. Which Measures Are Reasonable: Definition

- By a *non-additive measure*, we mean a function $\mu(A)$ that assigns, to each set $A \in \mathcal{A}$, a number $\mu(A) \geq 0$.
- A non-additive measure μ is *r-bounded* if it satisfies the following two properties:
  - for every $\Gamma > 0$, there exists a $\Delta > 0$ such that if $\mu(A) \leq \Gamma$ and $\mu(B) \leq \Gamma$, then $\mu(A \cup B) \leq \Delta$;
  - for every $\eta > 0$, there exists a $\nu > 0$ such that if $\mu(A) \leq \nu$ and $\mu(B) \leq \nu$ then $\mu(A \cup B) \leq \eta$.
- *Comment:* every additive measure is r-bounded, with $\Delta = 2 \cdot \Gamma$ and $\nu = \dfrac{\eta}{2}$.

6. Equivalence: Analysis of the Problem

- The numerical values of probabilities have observable sense.
- The probability of an event E can be defined as the limit of the frequency with which E occurs.
- In contrast, e.g., possibility values do not have direct meaning.
- The only important thing is which values are larger and which are smaller.
- This describes which events are more possible and which are less possible.
- From this viewpoint,
  - if two measures can be obtained from each other by a transformation that preserves the order,
  - such measures can be considered to be equivalent.

7. Definitions and the Main Result

- Non-additive measures $\mu(A)$ and $\mu'(A)$ are called *equivalent* if there exists a 1-1 monotonic function $f(x)$ s.t.

$$\mu'(A) = f(\mu(A)) \text{ for every } A.$$

- Let $\varepsilon > 0$ be a real number.
- A non-additive measure $\mu(A)$ is an *ε-possibility measure* if for every A and B:

$$\max(\mu(A), \mu(B)) \le \mu(A \cup B) \le (1+\varepsilon) \cdot \max(\mu(A), \mu(B)).$$

- *Result:* For every $\varepsilon > 0$, every r-bounded non-additive measure is equivalent to an ε-possibility measure.
- Can we strengthen this result? Is each r-bounded measure equivalent to a possibility measure?
- A simple answer is "No": any measure which is equivalent to a maxitive one is also maxitive.

8. First Auxiliary Result: Possibility of Uniform Equivalence

- We proved that *each* r-bounded non-additive measure can be re-scaled into an "almost" possibility measure.
- Sometimes, we have *several* measures.
- Can we re-scale all of them into ε-possibility measures by using the same re-scaling $f(x)$?
- *Result:* For every $\varepsilon > 0$, and for every finite set of r-bounded non-additive measures $\mu_1(A), \ldots, \mu_n(A)$,
  - there exists a 1-1 function $f(x)$ for which
  - all n measures $\mu'_i(A) \stackrel{\text{def}}{=} f(\mu_i(A))$ are ε-possibility measure.

9. Case of Generalized Metric

- Similarly to the fact that measures describe size, metrics describe distance.
- Usually, we considers metrics $d(a, b)$ which satisfy the triangle inequality $d(a, c) \le d(a, b) + d(b, c)$.
- However, it does not have to be this particular inequality. What is important is that:
  - if $d(a, b)$ and $d(b, c)$ are bounded by some $\Gamma > 0$,
  - then the distance $d(a, c)$ cannot be arbitrarily large,
  - it should be limited by some bound depending on the bound on $d(a, b)$ and $d(b, c)$.
- Similarly:
  - if $d(a, b)$ and $d(b, c)$ are sufficiently small,
  - then the distance $d(a, c)$ is also small.

10. Generalized Metric (cont-d)

- A function $d : X \times X \to R_0^+$ is called an *r-bounded metric* if it satisfies the following two properties:
  - for every $\Gamma > 0$, there exists a $\Delta > 0$ such that if $d(a,b) \le \Gamma$ and $d(b,c) \le \Gamma$, then $d(a,c) \le \Delta$;
  - for every $\eta > 0$, there exists a $\nu > 0$ such that if $d(a,b) \le \nu$ and $d(b,c) \le \nu$ then $d(a,c) \le \eta$.
- A function $d : X \times X \to R_0^+$ is called an *ultrametric* if

$$d(a,c) \le \max(d(a,b), d(b,c)) \text{ for all } a, b, \text{ and } c.$$

- $d : X \times X \to R_0^+$ is called an ε-*ultrametric* if

$$d(a,c) \le (1+\varepsilon) \cdot \max(d(a,b), d(b,c)) \text{ for all } a, b, \text{ and } c.$$

11. Generalized Metrics: Results

- r-bounded metrics $d(a,b)$ and $d'(a,b)$ are called *equivalent* if there exists a 1-1 monotonic $f(x)$ s.t.:

$$d'(a,b) = f(d(a,b)) \text{ for all } a, b.$$

- *Result:* For every $\varepsilon > 0$, every r-bounded metric is equivalent to an ε-ultrametric.
- For every $\varepsilon > 0$, and for every finite set of r-bounded metrics $d_1(a,b), \ldots, d_n(a,b)$, there is a 1-1 f-n $f(x)$ s.t.

$$d'_i(a,b) \stackrel{\text{def}}{=} f(d_i(a,b)) \text{ are } \varepsilon\text{-ultrametrics for all } i.$$

12. General Result Including Measures and Metrics as Special Cases

- By a *domain*, we mean a set S with a partial binary operation $\circ : S \times S \to S$.
- For measures, S is an algebra of sets, and $\circ$ is the union.
- For metrics, S is the set of all pairs (a, b), and the binary operation transforms (a, b) and (b, c) into (a, c).
- By a *characteristic*, we mean a function $F : S \to R_0^+$.
- A characteristic $F(x)$ is called *r-bounded* if:
  - $\forall \Gamma > 0\, \exists \Delta > 0$ s.t. if $F(x) \le \Gamma$, $F(x') \le \Gamma$, and $x \circ x'$ is defined, then $F(x \circ x') \le \Delta$;
  - for every $\eta > 0$, there exists a $\nu > 0$ such that if $F(x) \le \nu$ and $F(x') \le \nu$ then $F(x \circ x') \le \eta$.

13. General Results (cont-d)

- $F(x)$ is called an ε-*maxitive* if for all x and x' for which $x \circ x'$ is defined,

$$F(x \circ x') \le (1+\varepsilon) \cdot \max(F(x), F(x')).$$

- $F(x)$ and $F'(x)$ are called *equivalent* if there exists a 1-1 monotonic f-n $f(x)$ for which

$$F'(x) = f(F(x)) \text{ for all } x \in S.$$

- *Result:* for every $\varepsilon > 0$, every r-bounded characteristic is equivalent to an ε-maxitive one.

- *Result:* for every $\varepsilon > 0$, and for every finite set of r-bounded characteristics $F_1(x), \ldots, F_n(x)$,
  - there exists a 1-1 f-n $f(x)$ for which
  - all n characteristics $F'_i(x) \stackrel{\text{def}}{=} f(F_i(x))$ are ε-maxitive.

14. Conclusions

- The traditional probabilistic description of uncertainty uses additive probability measures.
- For describing non-probabilistic uncertainty, it is therefore reasonable to use non-additive measures.
- The most well-known example of such measures are possibility measures $\mu(A)$, for which:

$$\mu(A \cup B) = \max(\mu(A), \mu(B)) \text{ for all } A \text{ and } B.$$

- In this talk, we show that the wide use of possibility measures may be explained by the fact that:
  - under some reasonable conditions,
  - these measures can approximate any non-additive measures.

15. Conclusions (cont-d)

- Namely, for every $\varepsilon > 0$, each non-additive measure is isomorphic to an *ε-possibility measure*, s.t.:

  $$\max(\mu(A), \mu(B)) \le \mu(A \cup B) \le (1+\varepsilon) \cdot \max(\mu(A), \mu(B)).$$

- If we have several measures, then:
  - the tuple consisting of these measures
  - is isomorphic to a tuple of ε-possibility measures.
- Similar results are also proven for generalized metrics.

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17. Proof of the Main Result

- Let us first define a doubly infinite sequence $\ldots < c_{-1} < c_0 < c_1 < \ldots$ as follows.
- We take $c_0 = 1$; once we have defined c_k for some $k \geq 0$, we define c_{k+1} as follows.
- By def-n of an r-bounded measure, there exists $\Delta_k > 0$ s.t. if $\mu(A) \leq c_k$ and $\mu(B) \leq c_k$, then $\mu(A \cup B) \leq \Delta_k$.
- We then take $c_{k+1} \stackrel{\text{def}}{=} (1+\varepsilon) \cdot \max(c_k, \Delta_k)$.
- Here, $c_0 = 1$ and $c_{k+1} \geq (1+\varepsilon) \cdot c_k$; thus, $c_k \geq (1+\varepsilon)^k$ and $c_k \to \infty$ when k increases.
- Similarly, once we have defined the value c_{-k} for some $k \geq 0$, we define $c_{-(k+1)}$ as follows.
- By def-n of an r-bounded measure, there exists $\nu_k > 0$ s.t. if $\mu(A) \leq \nu_k$ and $\mu(B) \leq \nu_k$, then $\mu(A \cup B) \leq c_{-k}$.
- We then take $c_{-(k+1)} \stackrel{\text{def}}{=} (1-\varepsilon) \cdot \min(c_{-k}, \nu_k)$.

18. Proof (cont-d)

- Here, $c_0 = 1$ and $0 < c_{-(k+1)} \le (1-\varepsilon)\cdot c_{-k}$, hence $0 < c_{-k} \le (1-\varepsilon)^k$ and $c_{-k} \to 0$ when $k \to \infty$.
- Since $c_k \uparrow$, $c_k \to \infty$, and $c_{-k} \to 0$, for every $x > 0$, $\exists k$ s.t. $c_{k-1} < x \le c_k$; we define $f(x)$ as follows:
  - for each integer k, we take $f(c_k) = (1+\varepsilon)^{k/2}$ and
  - for each value x between c_{k-1} and c_k, we define $f(x)$ by linear interpolation: if $c_{k-1} < x \le c_k$, then

$$f(x) = f(c_{k-1}) + \frac{x - c_{k-1}}{c_k - c_{k-1}} \cdot (f(c_k) - f(c_{k-1})).$$

- Since the sequence c_k is strictly increasing, the resulting function $f(x)$ is also strictly increasing.
- W.l.o.g., we assume $\mu(A) \ge \mu(B)$.
- There exist integers k and ℓ for which $c_{k-1} < \mu(A) \le c_{k+1}$ and $c_{\ell-1} < \mu(B) \le c_\ell$; here, $k \ge \ell$ and $c_\ell \le c_k$.

19. Proof Finalized

- Hence, we have $\mu(A) \le c_k$ and $\mu(B) \le c_k$.
- By definition of Δ_k, we therefore have $\mu(A \cup B) \le \Delta_k$.
- By definition of c_{k+1}, this value is always greater than Δ_k, thence we have $\mu(A \cup B) \le c_{k+1}$.
- Since the function $f(x)$ is increasing, we get
$$\mu'(A \cup B) = f(\mu(A \cup B)) \le f(c_{k+1}) = (1+\varepsilon)^{(k+1)/2}.$$
- Here, $\max(\mu(A), \mu(B)) = \mu(A) > c_{k-1}$, so:
$$\max(\mu'(A), \mu'(B)) = \mu'(A) = f(\mu(A)) > f(c_{k-1}) = (1+\varepsilon)^{(k-1)/2},$$
$$\text{i.e., } (1+\varepsilon)^{(k-1)/2} < \max(\mu'(A), \mu'(B)).$$
- Multiplying both sides by $1+\varepsilon$, we get
$$(1+\varepsilon)^{(k+1)/2} < (1+\varepsilon) \cdot \max(\mu'(A), \mu'(B)).$$
- We already know that $\mu'(A \cup B) \le (1+\varepsilon)^{(k+1)/2}$.
- Thus, $\mu'(A \cup B) \le (1+\varepsilon) \cdot \max(\mu'(A), \mu'(B))$. Q.E.D.

20. Proof of the Auxiliary Result

- This proof is similar to the main result, the only difference is how we define c_{k+1} and $c_{-(k+1)}$.
- By definition of an r-bounded measure, for each i, there exists a value $\Delta_{ki} > 0$ for which:

$$\text{if } \mu_i(A) \le c_k \text{ and } \mu_i(B) \le c_k, \text{ then } \mu_i(A \cup B) \le \Delta_{ki}.$$

- We take $c_{k+1} = (1+\varepsilon) \cdot \max(c_k, \Delta_{k1}, \ldots, \Delta_{kn})$.
- By definition of an r-bounded measure, for each i, there exists a value $\nu_{ki} > 0$ for which:

$$\text{if } \mu_i(A) \le \nu_{ki} \text{ and } \mu_i(B) \le \nu_{ki}, \text{ then } \mu_i(A \cup B) \le c_{-k}.$$

- We take $c_{-(k+1)} = (1-\varepsilon) \cdot \min(c_{-k}, \nu_{k1}, \ldots, \nu_{kn})$.
- The rest of the proof is the same.