How to Estimate Relative Spatial Resolution of Different Maps or Images of the Same Area?

Christian Servin, Aaron Velasco, and Vladik Kreinovich

University of Texas at El Paso
El Paso, TX 79968, USA
cservin@mial.com, aavelasco@utep.edu,
vladik@utep.edu

1. Outline

- How to estimate relative spatial resolution of different maps or images of the same area under uncertainty?
- We consider probabilistic and fuzzy approaches.
- We show that both approaches lead to the same estimate.
- This makes us somewhat more confident that this joint result is reasonable.

2. Formulation of the Problem

- Different measurements results in maps of different spatial resolution.
- Example: geosciences.
  - gravity data are more accurate, but spatial resolution is low;
  - seismic data are less accurate, but have higher spatial resolution.
- Different techniques provide different pieces of information about the area.
- We would like to have a map that contains all this information.
- For that, we need to fuse the corresponding maps.
- To properly fuse the maps, we need to know their relative spatial resolution.

3. Need to Perform Estimations under Uncertainty

- We need to fuse images $\widetilde{I}_i(x)$, $i = 1, 2, \ldots$ corresponding to different spatial resolution.
- Spatial resolution refers to the relation between $\widetilde{I}_i(x)$ and the actual (ideal) image $I(x)$.
- In many cases, we have uncertainty: we do not know the relation between $\widetilde{I}_i(x)$ and $I(x)$.
- Traditionally, uncertainty in science and engineering is handled by the probabilistic approach.
- This approach originated in situations when we can experimentally determine the frequencies (probabilities).
- It is also actively used when we only have partial (or even no) information about these probabilities.

4. Probabilistic Approach: Examples

- Example 1:
  - if we have two alternatives and
  - we have no reason to assume that one of the them is more frequent than the other one,
  - then it makes sense to assume that these two alternatives have equal probabilities $1/2$.
- Example 2:
  - if we only know that a quantity x is located on an interval $[0, 1]$, and
  - we do not know which values from this interval are more probable or less probable,
  - it makes sense to assume that all these possible values are equally probable, i.e.,
  - that we have a uniform probability distribution on the interval $[0, 1]$.

5. Fuzzy Approach: Idea

- The probabilistic approach "makes up" the unknown probabilities based on common sense.
- Another idea is to explicitly *formalize* the common-sense ideas.
- *Problem:* commonsense ideas are often described by imprecise ("fuzzy") words from natural language.
- *Solution:* use *fuzzy techniques*, specifically developed to formalize natural-language knowledge.

6. Relation Between $\widetilde{I}_i(x)$ and $I(x)$: Probabilistic Approach

- Spatial uncertainty means that the value located at a point x is observed as corr. to $\widetilde{x} = x + \Delta x \approx x$.
- As a result, each value $I(x)$ gets distributed to values $I(x + \Delta x)$, for the corresponding random variable Δx.
- In general, there are many independent sources of spatial uncertainty.
- Δx can therefore be represented as a sum of many small independent random variables.
- Under reasonable assumptions, the distribution of such sums is close to Gaussian (*Central Limit Theorem*).
- So, we conclude that Δx is normally distributed.

7. Probabilistic Approach (cont-d)

- We conclude that Δx is normally distributed.
- In the isotropic case, the probability density is:

$$\rho(\Delta x) = \frac{1}{2\pi \cdot \sigma} \cdot \exp\left(-\frac{\|\Delta x\|^2}{2\sigma^2}\right).$$

- Each original value $I(x)$ is thus distributed, with this density, among the neighboring values.
- The observed $\widetilde{I}(y)$ can be obtained by adding the values $I(x) \cdot \rho(\Delta x)\, d\Delta x$ corresponding to $x + \Delta x = y$:

$$\widetilde{I}(y) = \int I(x) \cdot f(y - x)\, dx =$$

$$\text{const} \cdot \int I(x) \cdot \exp\left(-\frac{\|y - x\|^2}{2\sigma^2}\right)\, dx.$$

8. Relation Between $\widetilde{I}(x)$ and $I(x)$: Fuzzy Approach

- In fuzzy approach, we explicitly formalize the corresponding commonsense knowledge.
- The corresponding rules for each observed value $\widetilde{I}(y)$ are straightforward:

  If x is close to y, then $\widetilde{I}(y)$ is close to $I(x)$.

- Under reasonable assumptions, the way to describe closeness is by using a Gaussian membership function.
- In the isotropic case, we have

$$\mu(y-x)=\exp\left(-\frac{\|y-x\|^2}{2\sigma^2}\right).$$

- For each y, the value $\widetilde{I}(y)$ is equal to $I(x)$ with degree of membership $\mu(y-x)$.

9. Fuzzy Approach (cont-d)

- For each y, $\widetilde{I}(y)$ is $I(x)$ with degree $\mu(y-x)$, where
$$\mu(y-x) = \exp\left(-\frac{\|y-x\|^2}{2\sigma^2}\right).$$
- To transform this fuzzy information into a single (crisp) value, we can use, e.g., centroid defuzzification
$$\widetilde{I}(y) = \frac{\int I(x)\cdot\mu(y-x)\,dx}{\int \mu(y-x)\,dx}.$$
- The denominator is a constant not depending on y.
- Substituting the expression for the Gaussian membership function into this formula, we conclude that
$$\widetilde{I}(y) = \text{const}\cdot\int I(x)\cdot\exp\left(-\frac{\|y-x\|^2}{2\sigma^2}\right)\,dx.$$
- This is exactly the same formula as in the probability approach.

10. How to Estimate Relative Spatial Resolution: First Approximation

- Specifically, we have two images $\widetilde{I}_1(x)$ and $\widetilde{I}_2(x)$.
- According to our formulas, we have

$$\widetilde{I}_1(y) = C_1 \cdot \int I(x) \cdot \exp\left(-\frac{\|y-x\|^2}{2\sigma_1^2}\right)\,dx$$

$$\widetilde{I}_2(y) = C_2 \cdot \int I(x) \cdot \exp\left(-\frac{\|y-x\|^2}{2\sigma_2^2}\right)\,dx.$$

- We want to find the values σ_1 and σ_2.
- Formulas involving convolution are greatly simplified if we use Fourier transform:
  - if $h(y) = \int f(y) \cdot g(x-y)\,dx$,
  - then $H(\omega) = F(\omega) \cdot G(\omega)$.

11. First Approximation (cont-d)

- The Fourier transform $G(\omega)$ of a Gaussian $g(x) = \exp\left(-\frac{\|x\|^2}{2\sigma^2}\right)$ is $G(\omega) = \exp\left(-\frac{1}{2}\cdot\|\omega\|^2\cdot\sigma^2\right)$. Thus:

$$F_1(\omega) = C_1 \cdot F(\omega) \cdot \exp\left(-\frac{1}{2}\cdot\|\omega\|^2\cdot\sigma_1^2\right);$$

$$F_2(\omega) = C_2 \cdot F(\omega) \cdot \exp\left(-\frac{1}{2}\cdot\|\omega\|^2\cdot\sigma_2^2\right).$$

- From the above equations, one can conclude that

$$F_2(\omega) = C \cdot F_1(\omega) \cdot \exp\left(-\frac{1}{2}\cdot\|\omega\|^2\cdot(\sigma_2^2 - \sigma_1^2)\right).$$

- Thus, the only information about σ_i that we can extract from the maps is the difference $\Delta \stackrel{\text{def}}{=} \sigma_2^2 - \sigma_1^2$.
- It is reasonable to call this difference *relative spatial resolution* of the two images (maps).

12. A More Realistic Description

- In addition to spatial blurring, there is also inevitable additive noise (measurement error), so

$$\widetilde{I}_i(y) = C_1 \cdot \int I(x) \cdot \exp\left(-\frac{\|y-x\|^2}{2\sigma_i^2}\right)\, dx + n_i(y).$$

- As a result, for Fourier transforms, we get

$$F_i(\omega) = C_1 \cdot F(\omega) \cdot \exp\left(-\frac{1}{2} \cdot \|\omega\|^2 \cdot \sigma_i^2\right) + N_i(\omega).$$

- From these equations, we can conclude that

$$F_2(\omega) = C \cdot F_1(\omega) \cdot \exp\left(-\frac{1}{2} \cdot \|\omega\|^2 \cdot (\sigma_2^2 - \sigma_1^2)\right) + N(\omega),$$

where $C \stackrel{\text{def}}{=} \dfrac{C_1}{C_2}$ & $N(\omega) \stackrel{\text{def}}{=} N_2(\omega) - N_1(\omega) \cdot \exp\left(-\dfrac{1}{2} \cdot \|\omega\|^2 \cdot \Delta\right)$.

- This is a model that we will use to reconstruct the relative spatial resolution Δ.

13. How to Find C

- We have concluded that
$$F_2(\omega) = C \cdot F_1(\omega) \cdot \exp\left(-\frac{1}{2} \cdot \|\omega\|^2 \cdot (\sigma_2^2 - \sigma_1^2)\right) + N(\omega).$$
- The coefficient C can be found, e.g., by comparing the overall energy, i.e., by comparing the values for $\omega = 0$.
- For this value, we get $F_2(0) = C \cdot F_1(0) + N(0)$, so
$$C \approx \frac{F_2(0)}{F_1(0)}.$$
- Once C is estimated, we can divide $\widetilde{I}_2(x)$ (and thus, $F_2(\omega)$) by C, and get a simpler relation:
$$F_2(\omega) = F_1(\omega) \cdot \exp\left(-\frac{1}{2} \cdot \|\omega\|^2 \cdot \Delta\right) + N(\omega).$$

14. Taking Noise into Account

- In many practical cases, we do not know the exact characteristics of the additive noise.
- We only know noise order of magnitude n, such that $N(\omega) \approx n$.
- For frequencies ω for which $N_2(\omega) \approx n$, the whole observed value may be caused by noise.
- The corresponding values $N_i(\omega)$ do not carry any information about the actual image.
- Thus, they do not carry any information about Δ.
- So, we must only consider "above-noise" frequencies, for which $|F_2(\omega)| \geq c \cdot n$ for some constant $c \gg 1$.
- For these frequencies, we have

$$F_2(\omega) \approx F_1(\omega) \cdot \exp\left(-\frac{1}{2} \cdot \|\omega\|^2 \cdot \Delta\right) \text{ with accuracy } \approx n.$$

15. Taking Noise into Account (cont-d)

- The values of the Fourier transform are, in general, complex numbers.
- A complex number z can be characterized by its absolute value (modulus) $|z|$ and its phase.
- As one can see from the formulas, Δ does not affect the phases, so it is sufficient to consider absolute values:

$$|F_2(\omega)| \approx |F_1(\omega)| \cdot \exp\left(-\frac{1}{2} \cdot \|\omega\|^2 \cdot \Delta\right) \text{ with accuracy} \approx n.$$

- The above formula non-linearly depends on Δ.
- We can reduce this dependence to linear by using logarithms $\ell_i(\omega) \stackrel{\text{def}}{=} \ln(|F_i(\omega)|)$:

$$\ell_2(\omega) \approx \ell_1(\omega) - \frac{1}{2} \cdot \|\omega\|^2 \cdot \Delta \text{ with accuracy} \approx \frac{n}{|F_2(\omega)|}.$$

16. Resulting Formulas

- *Reminder:* for $\ell_i(\omega) = \ln(|F_i(\omega)|)$, we have:

$$\ell_2(\omega) \approx \ell_1(\omega) - \frac{1}{2} \cdot \|\omega\|^2 \cdot \Delta \text{ with accuracy} \approx \frac{n}{|F_2(\omega)|}.$$

- So, $\Delta \approx \dfrac{2(\ell_1(\omega) - \ell_2(\omega))}{\|\omega\|^2}$ with accuracy $\approx \dfrac{2n}{|F_2(\omega)| \cdot \|\omega\|^2}$.

- The Least Square Method for this problem leads to

$$\Delta = 2 \cdot \frac{\int (\ell_1(\omega) - \ell_2(\omega)) \cdot |F_2(\omega)|^2 \cdot \|\omega\|^2 \, d\omega}{\int |F_2(\omega)|^2 \cdot \|\omega\|^4 \, d\omega}.$$

- Here, integration is over frequencies ω for which $|F_2(\omega)| \geq c \cdot n$ for some pre-selected $c \gg 1$.

- *Preliminary results* show that this method correctly reconstructs the relative spatial resolution.

Formulation of the . . .
Need to Perform . . .
Probabilistic . . .
Fuzzy Approach: Idea
Probabilistic Approach
Fuzzy Approach
How to Estimate . . .
A More Realistic . . .
Resulting Formulas
Home Page
Title Page
◀◀ ▶▶
◀ ▶
Page 17 of 18
Go Back
Full Screen
Close
Quit

17. Acknowledgment

This work was also supported in part by the National Science Foundation grants

- HRD-0734825 and HRD-1242122 (Cyber-ShARE Center of Excellence), and
- DUE-0926721.