Fuzzy-Inspired Hierarchical Version of the von Neumann-Morgenstern Solutions as a Natural Way to Resolve Collaboration-Related Conflicts

Olga Kosheleva, Vladik Kreinovich, and
Martha Osegueda Escobar

University of Texas at El Paso
El Paso, TX 79968, USA
olgak@utep.edu, vladik@utep.edu,
mcoseguedaescobar@miners.utep.edu

1. Cooperative Games: Towards a Formal Description of Collaborative Situations

- Situations when all participants collaborate with each other are known as *cooperative games*.
- Let n denote the number of participants.
- For simplicity, the easier way to describe the participants is by simply numbering them.
- The set $\{1, \ldots, n\}$ of all n participants is usually denoted by N.
- When all the participants collaborate, as a single group N, they jointly gain some value $v(N)$.
- How to fairly divide this amount $v(N)$ between n participants?
- How to find an *imputation* $x = (x_1, \ldots, x_n)$ for which $x_i \geq 0$ and $\sum\limits_{i=1}^{n} x_i = v(N)$?

2. Cooperative Games (cont-d)

- Fair division means taking into account everyone's contribution:
  - if a group did not contribute anything, then it should not get much, and,
  - vice versa, if a group contributed almost everything, it should take home almost everything.
- So, we must take into account how much each group contributed.
- From the mathematical viewpoint, groups of individuals are *subsets* $S \subseteq N$ called *coalitions*.
- The contribution of each S can be described by the largest amount $v(S)$ that S could earn on its own.

3. It Makes Sense to Only Consider Gains Due to Collaboration

- Let us consider an excess over what individual can earn by themselves: $v'(S) \stackrel{\text{def}}{=} v(S) - \sum_{i \in S} v(\{i\})$.
- If disjoint coalitions S and S' collaborate, then they should be able to gain no less than on their own:

$$v(S \cup S') \geq v(S) + v(S').$$

- If S believes that in an imputation y, S will not get a fair share, S may force a switch to x, if:
    - x is within the reach of S, i.e., $\sum_{i \in S} x_i \leq v(S)$, and
    - all members of S gain more than in y: $x_i > y_i$ for all $i \in S$.
- We then say that x *dominates* y ($x \succ y$).

4. Solutions

- All this was introduced by John von Neumann (vN) and Oscar Morgenstern (M) in their pioneering book.
- Ideally, we should select a non-dominated imputation x; the set of all such x is called a *core*.
- *Problem:* not all games have cores.
- So, vN and M suggested to adopt a *social norm* C within which no two imputations dominate each other.
- The social norm has to be *enforceable*:
  - if someone proposes an imputation $x \notin C$,
  - then a coalition should force a switch to C.
- A set C is a *vN-M solution* if $x \not\succ y$ for $x, y \in C$ and

$$\forall y \notin C \, \exists x \in C \, (x \succ y).$$

- *Problem:* some games do not have NM-solutions at all.

5. Hierarchical Version of vN-M Solution

- When no NM-solution exists, we can still select an enforceable social norm C.
- For an NM-solution C, once we select an imputation $x \in C$, no switching is possible.
- In contrast, in the no-NM-solutions case, switching is still possible.
- A natural idea is to the further restrict imputations
  - first, we select the sets $C_1 \supset C_2 \supset C_3 \supset \ldots \supset C_k$;
  - then, we first force an imputation to be in C_1;
  - after that, we force the imputation to be in C_2, $\ldots$,
  - until we reach an imputation from the final set C_k in which no two imputations dominate each other.
- Here, we cannot enforce C_k in one step, but we can enforce C_k in several steps.

6. Hierarchical vN-M Solution (cont-d)

- *Example:* within the set C_1 of legal actions, there is a subset C_2 of ethical actions.
- There are several levels of ethical behavior – from not harming your neighbors C_2 to helping them C_k.
- *Relation to fuzzy:* C_i are actions which are ethical to a certain degree.
- By a *hierarchical vN-M solution*, we mean a finite sequence $C_0 = I \supset C_1 \supset C_2 \supset \ldots \supset C_k$ for which:
  - if $x, y \in C_k$, then $x \not\succ y$, and
  - for every $i \geq 0$, if $y \in C_i - C_{i+1}$, then there exists an $x \in C_{i+1}$ for which $x \succ y$.

7. First Result: Hierarchical vN-M Solutions Always Exist

- **Proposition 1.** *Every directed finite graph $(I, \succ)$ has a hierarchical von Neumann-Morgenstern solution.*
- **Proof.** We have the set $C_0 = I$. Let us inductively construct the desired sequence

$$C_0 \supset C_1 \supset \ldots \supset C_k.$$

- Let us assume that we already have constructed the sequence $C_0 \supset C_1 \supset \ldots \supset C_i$ for which:
  - for every $j \le i-1$ and for every $y \in C_j - C_{j+1}$,
  - there exists an $x \in C_{j+1}$ for which $x \succ y$.
- If $x \not\succ y$ for all $x, y \in C_i$, then $C_0 \supset C_1 \supset \ldots \supset C_i$ is a hierarchical NM-solution.
- If there are elements $x, y \in C_i$ for which $x \succ y$, then we can take $C_{i+1} = C_i - \{y\}$.

8. First Result (cont-d)

- If $x \succ y$ for some $x, y \in C_i$, then we take

$$C_{i+1} = C_i - \{y\}.$$

- The desired property of C_{i+1} is satisfied.
- Indeed, in this case, the only element y from $C_i - C_{i+1}$ is dominated by some element from C_i: namely, by x.
- At each step, we decrease the size of the set C_i.
- Since we started with a finite graph $C_0 = I$, this process will stop.
- We will then get the desired hierarchical vN-M solution.

9. Towards a Better Definition

- The above definition allows a huge number of layers.
- It is therefore desirable to decrease the number of such layers.
- One possibility is to require that for each set C_i:
  - the next set C_{i+1} should not be just a subset of C_i,
  - it should also be as small as possible.
- In other words:
  - in addition to the requirement that every $y \in C_i - C_{i+1}$ be dominated by some $x \in C_{i+1}$,
  - we should also require that we cannot have a smaller set $C'_{i+1} \subset C_{i+1}$ with this property.
- We will call such sequence a *strong hierarchical vN-M solution.*

10. Second Result

- **Proposition 2.** *Every directed finite graph $(I, \succ)$ has a strong hierarchical vN-M solution.*
- **Lemma.** *If a subset C of a finite graph $(I, \succ)$ contains two elements x and y for which $x \succ y$, then*
    - *there exists a set $C' \subset C$*
    - *which is a minimal next level for C.*
- **Proof of the Lemma.** We start with the set $C' = C - \{y\}$ which is a possible next level for C.
- If this set is minimal, we are done.
- If this set is not minimal, this means that:
    - there exists a subset $C'' \subset C'$ $(C'' \neq C')$
    - which is also a possible next level for C.
- If this set C'' is minimal, we are done.

11. Second Result (cont-d)

- If the set C'' is not minimal, this means that there exists an even smaller possible next level set C''', etc.
- We started with a finite set.
- We decrease the size by at least 1 on each iteration, so, eventually, we will find a minimal next level set.
- The lemma is proven.
- **Proof of the Proposition.**
- We start with the set $C_0 = I$.
- Once we have found the sets $C_1 \supset C_1 \supset \ldots \supset C_i$, if in C_i, there are no $x \succ y$, then we are done.
- If in the set C_i, there are connected elements, then we can use Lemma to find a minimal next level set for C_i.
- This set is what we take as C_{i+1}.

12. Second Result (final)

- At each step, we decrease the size of C_i, so this procedure will eventually stop.
- Thus, we will get the desired strong hierarchical vN-M solution.

13. Acknowledgments

This work was supported in part:

- by the National Science Foundation grants
  - HRD-0734825 and HRD-124212 (Cyber-ShARE Center of Excellence) and
  - DUE-0926721, and
- by an award from Prudential Foundation.