How to Transform Partial Order Between Degrees into Numerical Values

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1. Why Fuzzy Logic: A Brief Reminder

- In many practical situations, there are experts who are skilled in performing the corresponding task:
  - skilled machine operators successfully operate machinery,
  - skilled medical doctors successfully cure patients, etc.
- It is desirable to design automated systems that would help less skilled operators and doctors make proper decisions.
- It is important to incorporate the knowledge of the experts into these system.
- Some of this expert knowledge can be described in precise ("crisp") form.
- Such knowledge is relative easy to describe in precise computer-understandable terms.

2. Why Fuzzy Logic (cont-d)

- However, a significant part of human knowledge is described in imprecise ("fuzzy") terms like "small".
- One of the main objectives of fuzzy logic is to translate this knowledge into machine-understandable form.
- Zadeh proposed to describe, for each imprecise statement, a degree to which this statement is true.
- Intuitively, we often describe such degrees by using words from natural language, such as "very small".
- However, computers are not very good in precessing natural-language terms.
- Computers are more efficient in processing numbers.
- So, fuzzy techniques first translate the corresponding degrees into numbers from the interval $[0, 1]$.

3. Sometimes, the Corresponding Degrees Are Difficult to Elicit

- Some experts can easily describe their degrees in terms of numbers.
- Other experts are more comfortable describing degrees in natural-language terms.
- In this case, we need to translate the resulting terms into numbers from the interval $[0, 1]$.
- What information can we use for this translation?
- For some pairs of degrees, we know which degree corresponds to a larger confidence.
- For example, it is clear that "very small" is smaller than "somewhat small".
- It is reasonable to assume that these expert comparisons are transitive and cycle-free.

4. Towards Formulating the Problem

- Thus, we usually have a natural (partial) *order* relation between different degrees.
- This order is not necessarily total (linear): we may have two degrees with no relation between them, e.g.,
  - "reasonably small" and
  - "to some extent small".
- Thus, in general, this relation is a *partial order*.
- We would like to assign numbers from the interval [0, 1] to different elements from a partially ordered set.
- Of course, there are many such possible assignments.
- Our goal is to select the assignment which is, in some sense, the most reasonable.

5. Main Idea

- Let us number the elements of the original finite partially ordered set by numbers 1, 2, ..., k.
- Then we get the set $\{1, 2, \ldots, k\}$ with some partial order $\prec$.
- This order is, in general, different from the natural order $<$.
- The desired mapping means that we assign, to each of the numbers i from 1 to k, a real number $x_i \in [0, 1]$.
- In other words, we produce a tuple $x = (x_1, \ldots, x_k)$ of real numbers from the interval $[0, 1]$.
- The only restriction on this tuple is that if $i \prec j$, then $x_i < x_j$.
- Let us denote the set of all the tuples x that satisfy this restriction by $S_\prec$.

6. Main Idea (cont-d)

- Out of many possible tuples from the set $S_{\prec}$, we would like to select one $s = (s_1, \ldots, s_k)$.
- Which one should we select?
- Selecting a tuple means that we need to select, for each i, the corresponding value s_i.
- The ideally-matching tuple x has, in general, a different value $x_i \neq s_i$.
- It usually makes sense to describe the inaccuracy ("loss") of this selection by the square $(s_i - x_i)^2$.
- We do not know what is the ideal value x_i.
- We only know that this ideal value is the i-th component of some tuple $x \in S_{\prec}$.
- We have no reason to believe that some tuples are more probable than the others.

7. Main Idea (final)

- We have no reason to believe that some tuples are more probable than the others.
- As a result, it makes sense to consider them all equally probable.
- So, if we select the tuple s, then the expected loss is proportional to $\int_{S_\prec} (x_i - s_i)^2\,dx$.
- It is therefore reasonable to select a value s_i for which this loss is the smallest possible:

$$\int_{S_\prec} (x_i - s_i)^2\,dx \to \min_s .$$

8. From the Idea to an Algorithm

- Our objective is to come up with numbers describing expert degrees.
- So, we need a simple algorithm transforming a partial order into numerical values.
- Let us differentiate the objective function with respect to s_i and equate the resulting derivative to 0.
- As a result, we get $\int_{S_\prec} (s_i - x_i)\,dx = 0$, hence

$$s_i = \frac{N}{D}, \text{ where } N \stackrel{\text{def}}{=} \int_{S_\prec} x_i\,dx, \quad D \stackrel{\text{def}}{=} \int_{S_\prec} dx.$$

- Since $\prec$ is a *partial* order, we may have tuples $(x_1, \ldots, x_k)$ with different orderings between x_i.
- For example, if we know only that $1 \prec 2$ and $1 \prec 3$, then we can have $1 \prec 2 \prec 3$ and $1 \prec 3 \prec 2$.

9. From the Idea to an Algorithm (cont-d)

- In principle, we can also have equalities between x_i, but such have 0 volume.
- There are $k!$ possible linear orders ℓ between x_i.
- Let us denote the set of all the tuples with an order ℓ by T_ℓ.
- Then, each set $S_\prec$ is the union of the sets T_ℓ for all linear orders ℓ extending $\prec$: $S_\prec = \bigcup\limits_{\ell:\ell\supseteq\prec} T_\ell$.
- Thus, each of the integrals N and D over $S_\prec$ can be represented as the sum of integrals over the sets T_ℓ:

$$D = \sum_{\ell:\ell\supseteq\prec} D_\ell, \text{ where } D_\ell \stackrel{\text{def}}{=} \int_{T_\ell} x_i\,dx,$$

$$N = \sum_{\ell:\ell\supseteq\prec} N_\ell, \text{ where } N_\ell \stackrel{\text{def}}{=} \int_{T_\ell} dx.$$

10. From the Idea to an Algorithm (cont-d)

- Thus, to find s_i, it is sufficient to be able to compute the corresponding integrals.
- Each of these integrals can be computed by integrating variable-by-variable; for each variable x_j:
  - we integrate a polynomial with rational coefficients,
  - the ranges are between some values x_m and x_n,
  - so the integral is still a polynomial.
- After all integrations, we get a rational number.
- By adding D_ℓ and N_ℓ, we get D and N and, by dividing them, s_i.
- Actually, the value D_ℓ can be computed even faster: the integral D_ℓ is simply the volume of the set S_ℓ.
- The unit cube $[0, 1]^k$ of volume 1 is divided into $k!$ such parts of equal volume, so $D_\ell = \dfrac{1}{k!}$.

11. Example 1: A 1-Element Set

- Let us start with the simplest possible case, when we have a single degree.
- The partially ordered set has a single element 1.
- We want to find s_1.
- In this case, there is no order, so there are no restrictions on the values x_1.
- Thus, we have only one set T_ℓ which simply coincides with the interval $[0, 1]$.
- For this set, $D = 1$ and

$$N = \int_0^1 x_1 \, dx_1 = \frac{1}{2} \cdot x_1^2 \Big|_0^1 = \frac{1}{2}, \text{ thus, } s_1 = \frac{N}{D} = \frac{1}{2}.$$

- *Result:* in a situation when we know nothing about the degree, our idea leads to selecting $s_1 = 0.5$.

12. Example 2: A 2-Element Set With No Order

- Let us assume that we have two unrelated degrees 1 and 2.
- In this case, we can repeat the same argument for each of these sets and conclude that

$$s_1 = s_2 = \frac{1}{2}.$$

13. Example 3: An Ordered 2-Element Set

- Let us now consider the situation in which we have two ordered degrees: $1 \prec 2$.
- In this case, we need to compute two values $s_1 < s_2$ that correspond to these two degrees.
- In this situation, we have only one order ℓ: $1 \prec 2$.
- So, T_ℓ is the set of all the pairs (x_1, x_2) for which

$$x_1 < x_2.$$

- Thus, x_2 can take any value from the interval $[0, 1]$.
- Once x_2 is fixed, x_1 can take any value from 0 to x_2:

$$N_\ell = \int_{0\le x_1<x_{\le}1} x_1\,dx = \int_0^1 dx_2 \int_0^{x_2} x_1\,dx_1.$$

- The inner integral has the form

$$\int_0^{x_2} x_1\,dx_1 = \frac{1}{2}\cdot x_1^2\Big|_0^{x_2} = \frac{1}{2}\cdot x_2^2.$$

14. An Ordered 2-Element Set (cont-d)

- $N_\ell = \int_0^1 dx_2 \int_0^{x_2} x_1\,dx_1 = \int_0^1 dx_2 \cdot \frac{1}{2} x_2^2 = \frac{1}{6} \cdot x_2^3 \Big|_0^1 = \frac{1}{6}.$
- Here, $D_\ell = \frac{1}{2!} = \frac{1}{2}$, hence $s_1 = \frac{N}{D} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}.$
- For s_2: $N_\ell = \int_{0 \le x_1 < x_2 \le 1} x_2\,dx = \int_0^1 x_2 \cdot dx_2 \int_0^{x_2} dx_1.$
- We already know that the inner integral has the form $\int_0^{x_2} dx_1 = x_2$, thus

$$N_\ell = \int_0^1 x_2 \cdot dx_2 \int_0^{x_2} dx_1 = \int_0^1 x_2 \cdot dx_2 \cdot x_2 =$$

$$\int_0^1 x_2^2\,dx_2 = \frac{1}{3} \cdot x_2^3 \Big|_0^1 = \frac{1}{3}, \text{ and } s_2 = \frac{N}{D} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}.$$

15. General Case

- For a linearly ordered case $1 \prec 2 \prec \ldots \prec k$, we get $s_i = \dfrac{i}{k+1}$.
- So, in general, for a partial order $\prec$, the value s_i is equal to $s_i = \dfrac{r_i}{k+1}$, where:
  - r_i is the average value of the rank of the element i
  - in all the linear orders consistent with $\prec$.
- *Example:* when $1 \prec 2$, $1 \prec 3$, $1 \prec 4$, we have 6 orders:
  - an order in which $1 \prec 2 \prec 3 \prec 4$;
  - an order in which $1 \prec 3 \prec 4 \prec 2$;
  - an order in which $1 \prec 4 \prec 2 \prec 3$;
  - an order in which $1 \prec 4 \prec 3 \prec 2$;
  - an order in which $1 \prec 3 \prec 2 \prec 4$;
  - an order in which $1 \prec 2 \prec 4 \prec 3$.

16. Examples (cont-d)

- Here, 1 always has rank 1, so $r_1 = 1$, and $s_1 = \dfrac{1}{5}$.
- The average rank of each of the elements 2, 3, and 4 is $\dfrac{2+3+4+2+3+4}{6} = 3$, thus $s_2 = s_3 = s_4 = \dfrac{3}{5}$.
- *Example:* $1 \prec 2, \ldots, 1 \prec k$: $r_1 = 1$, so $s_1 = \dfrac{1}{k+1}$;

$$r_i = \frac{2+3+\ldots+k}{k-1} = \frac{\dfrac{k(k+1)}{2} - 1}{k-1} = \frac{k^2+k-2}{2(k-1)} =$$

$$\frac{(k+2)(k-1)}{2(k-1)} = \frac{k+2}{2}, \text{ so } s_i = \frac{k+2}{2(k+1)} = \frac{1}{2}\left(1 + \frac{1}{k+1}\right).$$

17. Interval-Valued Degrees

- Experts often have trouble providing an exact numerical degree.
- Indeed, we do not have a feeling of difference between, say, degree 0.5 and degree 0.501.
- From this viewpoint, it is more adequate to describe degrees
  - not by numbers but by intervals,
  - i.e., subintervals of the interval $[0, 1]$.
- It is therefore desirable to transform partial orders not into numbers, but into such intervals.
- Same idea works for interval-valued degrees.
- *Example:* two degrees $1 \prec 2$.
- We want to assign to each of them an interval $[\underline{s}_1, \overline{s}_1]$ and $[\underline{s}_2, \overline{s}_2]$.

18. Interval-Valued Degrees (cont-d)

- A natural way to describe that $1 \prec 2$ is to require that $\underline{s}_1 < \underline{s}_2$ and $\overline{s}_1 < \overline{s}_2$.
- Thus, we need to generate four numbers $\underline{s}_1$, $\overline{s}_1$, $\underline{s}_2$, and $\overline{s}_2$ for which $\underline{s}_1 < \overline{s}_1$, $\underline{s}_1 < \underline{s}_2$, $\overline{s}_1 < \overline{s}_2$, and $\underline{s}_2 < \overline{s}_2$.
- If we denote the corresponding bounds by 1^-, 1^+, 2^-, and 2^+, then we get the following partial order:

$$1^- \prec 1^+,\ 1^- \prec 2^-,\ 1^+ \prec 2^+, \text{ and } 2^- \prec 2^+.$$

- The only two degrees for which we have no ordering relation are 1^+ and 2^-.
- Thus, here we have two possible linear orders:
  - a linear order in which $1^- \prec 1^+ \prec 2^- \prec 2^+$, and
  - a linear order in which $1^- \prec 2^- \prec 1^+ \prec 2^+$.

19. Interval-Valued Degrees (cont-d)

- We have two possible linear orders:
  - a linear order in which $1^- \prec 1^+ \prec 2^- \prec 2^+$, and
  - a linear order in which $1^- \prec 2^- \prec 1^+ \prec 2^+$.
- Here, for the average ranks, we have

$$r_{1^-} = 1,\ r_{1^+} = r_{2^-} = \frac{2+3}{2} = 2.5, \text{ and } r_{2^+} = 4.$$

- Thus:

$$\underline{s}_1 = \frac{1}{5},\quad \overline{s}_1 = \underline{s}_2 = \frac{1}{2},\quad \overline{s}_2 = \frac{4}{5}.$$

- *Comment:* we can perform similar computations for any other partially ordered set.

20. Remaining Open Problems

- The above algorithm works OK.
- However, for a large number of degrees k, we may have exponentially many possible linear orders.
- This makes the computation of the average ranks r_i taking too much time.
- It is desirable to come up with a more efficient algorithm for computing the average ranks r_i:
  - by an appropriate Monte-Carlo method?
  - by an appropriate metaheuristic method?
- It is also desirable to extend the algorithm to cases:
  - when several experts describe different orders
  - when an expert is inconsistent (e.g., non-transitive).

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