Adding Possibilistic Knowledge to Probabilities Makes Many Problems Algorithmically Decidable

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1. Need to Supplement Probabilistic Predictions with Possibilistic Information

- Physical laws enable us to predict probabilities p.
- In general, probability p is a frequency f with which an event occurs, but sometimes, $f \neq p$.
- Example: due to molecular motion, a cold kettle on a cold stove can spontaneously boil with $p > 0$.
- However, most physicists believe that this event is simply not possible.
- This impossibility cannot be described by claiming that for some p_0, events with $p \leq p_0$ are not possible.
- Indeed, if we toss a coin many times N, we can get $2^{-N} < p_0$, but the result is still possible.
- So, to describe physics, we need to supplement probabilities with information on what is possible.

2. How to Describe Information about Possibility

- Let U be the universe of discourse, i.e., in our case, the set of possible events.
- We assume that we know the probabilities $p(S)$ of different events $S \subseteq U$.
- From all possible events, the expert select a subset T of all events which are possible.
- The main idea that if the probability is very small, then the corresponding event is not possible.
- What is "very small" depends on the situation.
- Let $A_1 \supseteq A_2 \supseteq \ldots \supset A_n \supseteq \ldots$ be a definable sequence of events with $p(A_n) \to 0$.
- Then for some sufficiently large N, the probability of the corresponding event A_N becomes very small.
- Thus, the event A_N is not impossible, i.e., $T \cap A_N = \emptyset$.

3. Resulting Definitions

- Let U be a set with a probability measure p.
- We say that $T \subseteq U$ is *a set of possible elements* if:
  - for every definable sequence A_n for which $A_n \supseteq A_{n+1}$ and $p(A_n) \to 0$,
  - there exists N for which $T \cap A_N = \emptyset$.
- Physicists uses a similar argument even when do not know probabilities.
- For example, they usually claim that:
  - when x is small,
  - quadratic terms in Taylor expansion $a_0 + a_1 \cdot x + a_2 \cdot x^2 + \ldots$ can be safely ignored.
- Theoretically, we can have a_2 s.t. $|a_2 \cdot x^2| \gg |a_1 \cdot x|$.
- However, physicists believe that such a_2 are not physically possible.

4. Definitions (cont-d)

- Physicists believe that very large values of a_2 are not physically possible.
- Here, we have $A_n = \{a_2 : |a_2| \geq n\}$.
- The physicists' belief is that for a sufficiently large N, event A_N is impossible, i.e., $A_N \cap T = \emptyset$.
- Here, $\cap A_n = \emptyset$, so $p(A_n) \to 0$ for any probability measure p.
- There are other similar conclusions, so we arrive at the following definition.
- We say that $T \subseteq U$ is *a set of possible elements* if:
  - for every definable sequence A_n for which $A_n \supseteq A_{n+1}$ and $\cap A_n = \emptyset$,
  - there exists N for which $T \cap A_N = \emptyset$.

5. In General, Many Problems Are Not Algorithmically Decidable

- A simple example is that it is impossible to decide whether two computable real numbers are equal or not.
- What are computable real numbers?
- In practice, real numbers come from measurements, and measurements are never absolutely accurate.
- In principle, we can measure a real number x with higher and higher accuracy.
- For any n, we can measure x with accuracy 2^{-n}, and get a rational r_n for which $|x - r_n| \leq 2^{-n}$.
- A real number is called computable if there is a procedure that, given n, returns x_n.

6. Many Problems Are Not Algorithmically Decidable (cont-d)

- Computing with computable real numbers means that,
  - in addition to usual computational steps,
  - we can also, given n, ask for r_n.
- Some things can be computed: e.g., given x and y, we can compute $z = x + y$.
- However, it is not possible to algorithmically check whether $x = y$.
- Indeed, suppose that this was possible.
- Then, for $x = y = 0$ with $r_n = s_n = 0$ for all n, our procedure will return "yes".
- This procedure consists of finitely many steps, thus it can only ask for finitely many values r_n and s_n.

7. Many Problems Are Not Algorithmically Decidable (cont-d)

- The $x \stackrel{?}{=} y$ procedure consists of finitely many steps, thus it can only ask for finitely many values r_n and s_n.
- Let N be the smallest number which is larger than all such requests n. So:
  - if we keep $x = 0$ and take $y' = 2^{-N} \neq 0$ with $s'_1 = \ldots = s'_{N-1} = 0$ and $s'_N = s'_{N+1} = \ldots = 2^{-N}$,
  - our procedure will not notice the difference and mistakenly return "yes".
- This proves that a procedure for checking whether two computable numbers are equal is not possible.
- Similar negative results are known for many other problems.

8. Under Possibility Information, Equality Becomes Decidable: Known Result

- On the set $U = \mathbb{R} \times \mathbb{R}$ of all possible pairs of real numbers, we have a subset T of possible numbers.
- In particular, we can consider the following definable sequence of sets $A_n \stackrel{\text{def}}{=} \{(x, y) : 0 < |x - y| \le 2^{-n}\}$.
- One can easily see that $A_n \supseteq A_{n+1}$ for all n and that $\cap A_n = \emptyset$.
- Thus, there exists a natural number N for which no element $s \in T$ belongs to the set A_N.
- This, in turn, means that for every pair $(x, y) \in T$, either $|x - y| = 0$ (i.e., $x = y$) or $|x - y| > 2^{-N}$.
- So, to check whether $x = y$ or not, it is sufficient to compute both x and y with accuracy $2^{-(N+2)}$.

9. Under Possibilistic Information, Many Problems Become Decidable: A New Result

- In terms of sequences r_n and s_n, equality $x = y$ can be described as $\forall n\,(|r_n - s_n| \le 2^{-(n-1)})$.
- Many properties involving limits, differentiability, etc., can be described by *arithmetic formulas*

$$\Phi \stackrel{\rm def}{=} Qn_1\, Qn_2 \ldots Qn_k\, F(r_1, \ldots, r_\ell, n_1, \ldots, n_k).$$

- Here, Qn_i is $\forall n_i$ or $\exists n_i$; $r_1, \ldots, r_\ell$ are sequences.
- F is a propositional combination of $=$'s and $\ne$'s between computable rational-valued expressions.
- For every Φ, for every set T of possible tuples $r = (r_1, \ldots, r_\ell)$, there exists an algorithm that,
  - given a tuple $r = (r_1, \ldots, r_\ell) \in T$,
  - checks whether Φ is true.

10. Proof by Quantifier Elimination

- We show that an expression $\exists n_i\, G(n_i)$ or $\forall n_i\, G(n_i)$ is equivalent to a quantifier-free formula.
- Here, $\exists n_i\, G(n_i) \Leftrightarrow \neg\forall n_i\, \neg G(n_i)$, so it is sufficient to prove it for $\forall$.
- Then, by eliminating quantifiers one by one, we get an equivalent easy-to-check quantifier-free formula.
- Take $A_n = \{r : \forall n_1\, (n_1 \le n \to G(n_1))\, \&\, \neg\forall n_1\, G(n_1)\}$.
- One can easily check that $A_n \supseteq A_{n+1}$ and $\cap A_n = \emptyset$.
- Thus, there exists N for which $T \cap A_N = \emptyset$.
- So, for $r \in T$, if $\forall n_1\, (n_1 \le N \to G(n_1))$, we cannot have $\neg\forall n_1\, G(n_1)$, so we must have $\forall n_1\, G(n_1)$.
- Thus, for $r \in T$, $\forall n_1\, G(n_1)$ is equivalent to a quantifier-free formula $G(1)\, \&\, G(2)\, \&\, \ldots\, \&\, G(N)$.

11. Relation to Possibility Theory

- Physics is versatile, and it is important to have several experts to cover all possible topics.
- Let E denote the set of all the experts.
- Experts, in general, may have somewhat different ideas on what is possible and what is not.
- For each event s, we have a set $m(s) \subseteq E$ of all the experts who believe that s is possible.
- For each set of events $S \subseteq U$, S is possible if one of $s \in S$ is possible, so $m(S) = \bigcup\limits_{s \in S} m(s)$.
- Thus, for every S and S', we have

$$m(S \cup S') = m(S) \cup m(S').$$

- So, we have a possibility measure m describing what physicists believe to be possible.

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