

From Fuzzy Universal Approximation to Fuzzy Universal Representation: It All Depends on the Continuum Hypothesis

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1. Need to Translate Expert Statements into Precise Terms

- In many practical situations,
 - there is a correlation between two quantities x and y ,
 - the only information that we have to describe this correlation are expert statements, and
 - these statement use imprecise (“fuzzy”) natural language words, such as “small”.
- For example, an expert can say that if x is small, then y is big, and vice versa.

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2. Fuzzy Logic Provides the Desired Translation

- Fuzzy logic is a technique that translates this knowledge into precise mathematical terms.
- In this technique, each fuzzy term A is described by a function $A(x)$ assigning,
 - to each possible value x of the corresponding quantity,
 - a degree $A(x)$ to which this value has the appropriate property (e.g., is small).
- The expert gives us rules $A_i(x) \Rightarrow B_i(y)$.
- The degree $d(x, y)$ to which each pair (x, y) is possible can be described as the degree to which:
 - either the first rule is satisfied (i.e., $A_1(x)$ is true and $B_1(y)$ is true)
 - or the second rule is satisfied, etc.

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3. Fuzzy Logic (cont-d)

- The degree $d(x, y)$ to which each pair (x, y) is possible can be described as the degree to which:
 - either the first rule is satisfied (i.e., $A_1(x)$ is true and $B_1(y)$ is true)
 - or the second rule is satisfied, etc.
- One possible way to interpret “and” is to use product, and “or” is sum.
- Then, the desired degree takes the form

$$d(x, y) = \sum_{i=1}^n A_i(x) \cdot B_i(y).$$

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4. Formulation of the Problem

- Fuzzy rules lead to the formula

$$d(x, y) = \sum_{i=1}^n A_i(x) \cdot B_i(y).$$

- It is known that the above expression has a universal approximation property:
 - for every $\varepsilon > 0$,
 - every continuous function on a box can be ε -approximated by such sums.
- Natural question:* when can we get an *exact* representation of every function?

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5. Let us Formulate the Problem in Precise Terms: First Attempt

- We start with the formula

$$d(x, y) = \sum_{i=1}^n A_i(x) \cdot B_i(y).$$

- The simplest way to interpret the above question is to ask whether there exists an integer n such that
 - *any* function of two variables can be represented in this form with this particular n .
- It turns out that this is not possible: this was proven by Roy O. Davies from Purdue in 1974.

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6. Let us Formulate the Problem in Precise Terms: Second Attempt

- We cannot have universal representation by using a fixed finite number of terms.
- A natural next idea is to have a representation in which:
 - the number of terms is finite for every function $d(x, y)$ and for every pair (x, y) , but
 - this number of terms may be different for different functions $d(x, y)$ and different pairs (x, y) .
- Thus, we arrive at the following definition.

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7. Definition and the Resulting Question

- **Definition.** *We say that there is a universal representation property if:*

- *every function $d(x, y)$ of two variables*
- *can be represented as the sum*

$$d(x, y) = \sum_{i=1}^{\infty} A_i(x) \cdot B_i(y),$$

- *so that for every pair (x, y) , only finitely many terms in the sum are different from 0.*
- *Resulting question:* is there a universal representation property?

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8. Somewhat Surprising Answer: It All Depends on the Continuum Hypothesis

- Intuitively, we expect:
 - either a positive answer to the above question – i.e., a proof that the universal representation is possible,
 - or a proof that such a universal representation is not possible.
- The actual answer is not what we would intuitively expect.
- **Proposition 1.** *The universal representation property is equivalent to the Continuum Hypothesis.*
- This was also proved by Roy O. Davies in 1974.
- So, the universal representation property depends on a somewhat obscure hypothesis from set theory.

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9. What is the Continuum Hypothesis: Reminder

- In the late 19th century, Georg Cantor invented set theory.
- Set theory is now the foundations of mathematics.
- He proved that each infinite subset S of the set N of natural numbers is equivalent to N – in the sense that:
 - there is a 1-to-1 correspondence
 - between N and S .
- He also proved that the *continuum* – i.e., the set R of real numbers – is *not* equivalent to N .
- Cantor conjectured that every infinite subset S of the continuum R is equivalent either to N or to R .
- This conjecture became known as *Continuum Hypothesis* (CH).

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10. Continuum Hypothesis (cont-d)

- Working mathematicians usually assume this hypothesis.
- But can this hypothesis can be proven or disproven based on other – more intuitive – axioms?
- This remained an open problem for a long time.
- The first breakthrough came from the famous logician Kurt Gödel.
- Gödel proved, in 1940, that the negation of CH cannot be proven in set theory.
- He proved it by showing that:
 - if set theory is consistent, i.e., has a model,
 - then, based on this model, we can build another model in which CH is true.

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11. Continuum Hypothesis (final)

- Can CH or its negation be derived from other axioms of set theory?
- This question was settled in the 1960s, when Paul Cohen proved that CH is independent of set theory.
- In other words, we can neither prove nor disprove it based on other axioms of set theory.
- For this result, Paul Cohen was awarded the Fields Medal – the math. equivalent of the Nobel Prize.

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12. Why This Result is Interesting

- At first glance:
 - the Continuum Hypothesis is an obscure statement of set theory,
 - of little interest to working mathematicians
 - and probably of even less interest to applications of mathematics.
- However, surprisingly, this abstract statement is equivalent to something much more practical.
- Namely, CH is equivalent the universal representation property for fuzzy systems.

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13. Why This Result is Interesting (cont-d)

- Of course, one can argue that
 - in practice, when everything is measured and implemented with some accuracy anyway,
 - all we care about is the universal *approximation* property.
- However, the universal representation property also makes application sense.
- This property shows that we can have an approximation in which:
 - for every property $d(x, y)$ and for every pair (x, y) ,
 - the number of non-zero terms remains constant no matter how much accuracy we seek;
 - thus, the number of expert rules remains constant when we increase the accuracy.

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16. Acknowledgments

This work was supported in part:

- by the National Science Foundation grants:
 - HRD-0734825 and HRD-1242122
(Cyber-ShARE Center of Excellence) and
 - DUE-0926721, and
- by an award from Prudential Foundation.

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