Fuzzy Sets As Strongly Consistent Random Sets

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1. Predictions Are Important

- One of the main applications of science and engineering:
  - is to predict future events, and
  - for engineering, to come up with designs and controls for which the future is the most beneficial.
- For example:
  - science predicts the position of the Moon in a few months, while
  - engineering predicts the position of the spaceship in a month, and
  - describes the best trajectory correction.

2. Predictions Are Important (cont-d)

- Some scientists say – correctly – that the main objective of science is to explain the world.
- But what does this mean in practical terms?
- How can we prove prove that a new physical theory explains the world better?
- By showing that it enables us to give more accurate predictions of future events.
- This is how General Relativity became accepted:
  - when experiments confirmed its prediction of
  - how much the light ray passing near the Sun will be distorted by the Sun's gravitational field.

3. Perfect Knowledge is Rarely Available: Need for Set Uncertainty

- Usually, we have only partial knowledge.
- Thus, instead of a single future state, we have a *set* of future states.
- One way to predict the future state is to look for similar situations in the past.
- In the case of partial knowledge, we may have several different similar situations in the past.
- We can predict that the future situation will be similar to one of the corresponding outcomes.

4. From Set Uncertainty to Probabilistic Uncertainty

- When we have many similar situations,
  - we can determine not only which future states are possible,
  - but also how frequent are different future states.
- For each of the possible future states $s_1, \ldots, s_n$:
  - the observed frequency of this state
  - serves as a natural estimate for the *probability* p_i of this state.
- Thus, in this case:
  - we know the set of possible states $s_1, \ldots, s_n$, and
  - we know the probabilities $p_1, \ldots, p_n$ of different possible states, probabilities adding up to 1: $\sum_{i=1}^{n} p_i = 1$.

5. From Probabilistic Uncertainty to Random Set Uncertainty

- Past observations were also only partial.
- We did not get a full knowledge of a state, we get a partial knowledge.
- So, each past observations $o_1, \ldots, o_n$ corresponds:
  - not to a single state, but
  - to the whole set S_i of possible states.
- Example: a measuring instrument with accuracy 0.1 records:
  - the value 1.0 in 40% of the cases,
  - the value 1.1 in 20% of the cases, and
  - the value 1.2 in the remaining 20% of the cases.

6. Random Set Uncertainty (cont-d)

- This means that:
  - with probability 40%, we have values from the interval $S_1 = [0.9, 1.1]$;
  - with probability 20%, we have values from the interval $S_2 = [1.0, 1.2]$; and
  - with the remaining probability 20%, we have values from the interval $S_3 = [1.1, 1.3]$.
- A situation in which we have several sets with different probabilities is known as a *random set.*
- This is similar to how:
  - the situation when we have different numbers with different probabilities is a random number, and
  - the situation when we have different vectors with different probabilities is known as a random vector.

7. We Will Consider Finite Sets

- In practice:
  - because of the limits of measurement accuracy,
  - only finitely many different states are distinguishable.
- For example:
  - even if we can measure lengths from 0 to 1 m with accuracy of 1 micron,
  - we still have only a million possible values.
- Thus, in this paper, we will assume that our Universe of discourse U is finite.
- Once a finite set U is fixed, we can define a *random set* as a set of pairs (S_i, p_i), where $S_i \subseteq U$, $p_i > 0$, and

$$\sum_{i=1}^{n} p_i = 1.$$

8. Relation to Fuzzy Sets

- A fuzzy set, for each possible state $x \in U$, describes the degree $\mu(x)$ to which this state x is possible.
- We can gauge this degree by the probability that x is possible w.r.t. the corr. observation S_i (i.e., $x \in S_i$):

$$\mu(x) = \sum_{i: x \in S_i} p_i.$$

- Every membership function can be thus interpreted: sort the values $\mu(x_1), \ldots, \mu(x_n)$ in a decreasing order:

$$\mu(x_{(1)}) \geq \mu(x_{(2)}) \geq \ldots \geq \mu(x_{(n)}); \text{ then:}$$

  - $S_0 = \emptyset$ with prob. $p_0 = \mu(x_{(1)})$;
  - $S_1 = \{x_{(1)}\}$ with prob. $p_1 = \mu(x_{(1)}) - \mu(x_{(2)})$;
  - $S_k = \{x_{(1)}, \ldots, x_{(k)}\}$, $p_k = \mu(x_{(k)}) - \mu(x_{(k+1)})$.
- Here, $\sum_{i=1}^{n} p_i = 1$ and $\sum_{i: x_{(k)} \in S_i} p_i = \mu(x_{(k)})$ for all k.

9. Relation to Fuzzy Sets (cont-d)

- For a *normalized* fuzzy set, for which $\max\limits_k \mu(x_k) = 1$, there is no need for a weird empty set.
- There are other possible random sets that lead to the same fuzzy set μ.
- As a result, we can interpret a fuzzy set $\mu(x)$ as an equivalence class of random sets.
- Namely, a fuzzy set is the class of all random sets for which, for every $x \in U$, we have $\sum\limits_{i: x \in S_i} p_i = \mu(x)$.
- In general, a random set is nothing else but a mass distribution in the Dempster-Shafer approach, then

$$\mu(x) = \mathrm{Pl}(\{x\}).$$

10. Current Interpretation of Fuzzy Sets in Terms of Random Sets: Advantages and Limitations

- The above interpretation:
  - helps to teach fuzzy techniques to statisticians and
  - enables us to apply results about random sets to fuzzy techniques.
- The main problem with this interpretation is that it is too complicated.
- A random set is not an easy notion, and classes of random sets are even more complex.
- This complexity goes against the spirit of fuzzy sets, whose purpose was to be simple and intuitively clear.
- From this viewpoint, it is desirable to simplify this interpretation.

11. What We Do in This Paper

- We show that fuzzy sets can be interpreted
  - not as classes, but as
  - *strongly consistent* random sets (in some reasonable sense).
- This is not yet at the desired level of simplicity.
- However, this new interpretation is much simpler than the original one.
- It thus constitutes an important step towards the desired simplicity.

12. Notion of Consistency

- Often, different alternative are inconsistent and thus, different sets S_i and S_j are disjoint: $S_i \cap S_j = \emptyset$.
- Example: for accuracy 1.1, measurement results 1.0 and 1.3 are inconsistent: $[0.9, 1.1] \cap [1.2, 1.4] = \emptyset$.
- Sometimes, we have *consistency*: every two sets S_i and S_j have a non-empty intersection.
- This is true, e.g., for the random set that we used to represent a given fuzzy set.
- This is also true for the above example of three measurements 1.0, 1.1, and 1.2, with sets

$$[0.9, 1.1], \quad]1.0.1.2], \quad [1.1, 1.3]$$

- Let us require that the random set be consistent.

13. Need for Strong Consistency

- Sometimes, we learn an additional information, e.g., we learn that some alternative x is not possible.
- In this case, a previously consistent random set may stop being consistent.
- Example: a random set with $S_1 = \{x_1, x_2\}$, $S_2 = \{x_2, x_3\}$, and $p_1 = p_2 = 0.5$ is consistent:

$$S_1 \cap S_2 = \{x_2\} \neq \emptyset.$$

- However, if we learn that x_2 is not possible, it stops being consistent: for $S_1' = \{x_1\}$ and $S_2' = \{x_3\}$:

$$S_1' \cap S_2' = \emptyset.$$

- It is reasonable to require that the random set remain consistent when we learn additional information.
- We will call such random sets *strongly consistent*.

14. How Does a Random Set Change When We Learn Additional Information

- Suppose that we had the Universe of discourse U.
- Then we learn that only some of the original alternatives are possible.
- Let $S \subseteq U$ denote the set of all possible alternatives; then: $p(S'_i \mid S_i \text{ is possible}) = \dfrac{p_i}{\sum\limits_{j:S_j \cap S \neq \emptyset} p_j}$.
- In Dempster-Shafer terms, the denominator is equal to the plausibility $\mathrm{Pl}(S)$ of the set S.
- Some of the sets may become equal, so we will have to combine their probabilities: $p'(s) = \dfrac{\sum\limits_{i:S_i \cap S = s} p_i}{\sum\limits_{i:S_i \cap S \neq \emptyset} p_i}$.

15. Definitions

- Let U be a finite set; we will call this set the *Universe of discourse.*
- By a *random set,* we mean a pair $((S_1,\ldots,S_n),(p_1,\ldots,p_n))$, where
$$S_i \in U, \quad p_i > 0, \text{ and } \sum_{i=1}^{n} p_i = 1.$$
- A *fuzzy set* μ is a function from U to $[0,1]$.
- A fuzzy set $\mu(x)$ is called *normalized* if $\max_x \mu(x) = 1$.
- We say that a fuzzy set $\mu(x)$ is *consistent* with a random set $((S_1,\ldots,S_n),(p_1,\ldots,p_n))$ if for every $x \in U$:
$$\mu(x) = \sum_{i: x \in S_i} p_i.$$

16. Definitions (cont-d)

- By a *standard random set* $\mathcal{S}_\mu$ *corresponding to the fuzzy set*, we mean that following random set:
  - we sort the values $\mu(x)$ into the decreasing sequence $\mu(x_{(1)}) \ge \ldots \ge \mu(x_{(n)})$, and
  - we take $S_i = \{x_{(1)}, \ldots, x_{(i)}\}$ with $p_i = \mu(x_{(i)}) - \mu(x_{(i+1)})$ for $i < n$ and $p_n = \mu(x_{(n)})$.
- We say that a random set $((S_1, \ldots, S_n), (p_1, \ldots, p_n))$ is *consistent* if $S_i \cap S_j \neq \emptyset$ for all i and j.
- A set $S \subseteq U$ is *consistent* with a random set $\mathcal{S} = ((S_1, \ldots, S_n), (p_1, \ldots, p_n))$ if $S \cap S_i \neq \emptyset$ for some i.

17. Definitions (final)

- If S is consistent with $\mathcal{S}$, we can define the restriction $\mathcal{S}_S$ as follows:
  - it has non-empty sets s of the type $S_i \cap S$
  - with probabilities $p'(s) = \dfrac{\sum\limits_{i: S_i \cap S = s} p_i}{\sum\limits_{i: S_i \cap S \neq \emptyset} p_i}$.
- We say that a random set is *strongly consistent* if all its restrictions are consistent.
- One can easily see that the standard random set corresponding to a fuzzy set is strongly consistent.
- Interestingly, as our result shows, the inverse is also true.

18. Main Result

- **Proposition.** *For every normalized fuzzy set $\mu(x)$:*
  - *the only strongly consistent random set consistent with $\mu(x)$*
  - *is the standard random set $\mathcal{S}_\mu$.*
- One of the problems of the existing random-set interpretation of fuzzy sets is that in this interpretation:
  - each fuzzy set $\mu(x)$ is associated with
  - the whole *class* of random sets.
- If we restrict ourselves to strongly consistent $\mathcal{S}$, then to each $\mu(x)$ there corresponds a *unique* random set.
- Thus, the interpretation of fuzzy sets as strongly consistent random sets is indeed simpler.

19. Proof: Main Idea

- Let us prove that if $\mathcal{S}$ is strongly consistent, then for every two S_i and S_j, either $S_i \subseteq S_j$ or $S_j \subseteq S_i$.
- We will prove this by contradiction.
- Let us assume that for some i and j, we have $S_i \not\subseteq S_j$ and $S_j \not\subseteq S_i$, i.e., $S_i - S_j \neq \emptyset$ and $S_j - S_i \neq \emptyset$.
- Let us then consider the set $S = U - (S_i \cap S_j)$.
- This set S is consistent with $\mathcal{S}$, since

$$S \cap S_i = S_i - S_j \neq \emptyset \text{ and } \cap S_j = S_j - S_i \neq \emptyset.$$

- Thus, in the restriction $\mathcal{S}_S$, we have elements

$$S_i' = S \cap S_i = S_i - S_j \text{ and } S_j' = S \cap S_j = S_j - S_i.$$

- One can easily see, however, that the sets S_i' and S_j' do not have any common elements.

20. Proof (cont-d)

- We have $S'_i \cap S'_j = \emptyset$.
- This contradicts to our assumption that $\mathcal{S}$ is strongly consistent.
- The contradiction proves that the sets S_i are indeed linearly ordered by inclusion.
- Thus, by a 2014 result by M. Daniel, $\mathcal{S}$ is the standard random set associated with the given fuzzy set.

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