Scaling-Invariant Description of Dependence Between Fuzzy Variables: Towards a Fuzzy Version of Copulas

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1. Fuzzy Degrees: a Brief Reminder

- In many real-life situations, it is important to incorporate expert knowledge into a computer-based system.
- Experts are often not 100% confident about their statements.
- They may use heuristic rules that they know to be sometimes false.
- Thus, it is important to describe the expert's degree of confidence in different statements.
- Experts usually describe their degree of confidence by using words from a natural language, such as "usually".
- However, computers are not very efficient in processing natural language.
- They are more efficient in doing what they were originally designed for – processing numbers.

2. Fuzzy Degrees (cont-d)

- It is therefore reasonable to describe expert's degrees of confidence by numbers.
- In the computers, "true" is usually represented as 1, and "false" as 0.
- Thus, it makes sense to represent intermediate degrees of confidence by numbers from the interval $[0, 1]$.
- This is one of the main ideas behind fuzzy logic.
- There are many ways to assign a numerical degree to a natural-language term.
- For example, we can ask an expert to mark his/her degree of confidence on a scale from, say, 0 to 10.
- If the expert marks 7, we take the ratio $7/10$ as the desired degree of confidence.

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3. Fuzzy Degrees (cont-d)

- Alternatively, we can ask the expert to select between:
  - getting a certain small monetary award when his/her statement is true in a random situation or
  - getting the same award with some probability.
- This measures the expert's subjective probability that his/her statement is true.
- In general, different methods lead to different numerical degrees.
- In all these cases, the more confident the expert is in a statement, the larger the numerical degree.
- Thus, ideally, a larger degree on one scale corresponds to a larger number on a different scale.

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4. Formulation of the Problem

- Often, a term used by an expert depends on two or more real-valued variables.
- Example: healthiness degree depends on blood pressure, body-mass index, etc.
- For each combination $(x_1, \ldots, x_n)$, we have a degree $\mu(x_1, \ldots, x_n) \in [0, 1]$.
- The membership function $\mu(x_1, \ldots, x_n)$ described the dependence between x_i.
- However, the numerical values of this function change if we use a different scale.
- In this talk, we describe a scale-invariant way to describing this dependence.

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5. Copulas Solve a Similar Prob. Problem

- A random variable X is described by its cumulative distribution function (cdf) $F(x) \stackrel{\text{def}}{=} \text{Prob}(X \le x)$.
- A joint distribution is described by a joint cdf

$$F(x_1, x_2) \stackrel{\text{def}}{=} \text{Prob}(X_1 \le x_1 \,\&\, X_2 \le x_2).$$

- For each i, we have a *marginal* $F_i(x_i) \stackrel{\text{def}}{=} \text{Prob}(X_i \le x_i)$:

$$F_1(x_1) = F(x_1, +\infty) \text{ and } F_2(x_2) = F(+\infty, x_2).$$

- The joint cdf contains the information:
  - about the marginals and
  - about the relation between X_i.
- How can we describe just the information about the dependence between the random variables?

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6. Copulas (cont-d)

- E.g., independence means $F(x_1, x_2) = F_1(x_1) \cdot F_2(x_2)$.
- In general, $F(x_1, x_2) = C(F_1(x_1), F_2(x_2))$ for some function $C(u_1, u_2)$ called a *copula*.
- For $n > 2$ variables,

$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)).$$

- Copulas do not change if we re-scale the variables x_i:

$$x_i \to x_i' = f_i(x_i).$$

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7. What Is the Fuzzy Analog of a Marginal?

- We want to use an analogy with copulas.
- For each pair (x_1, x_2), the value $\mu(x_1, x_2)$ describes the degree to which this pair is possible.
- A value x_1 is possible if (x_1, x_2) is possible for some x_2:
  - either $(x_1, 0)$ is possible,
  - or $(x_1, 0.01)$ is possible, etc,
- To get a degree to which x_1 is possible, we need to combine the degrees $\mu(x_1, x_2)$ by some "or"-operation.
- In principle, there are many different "or"-operations: $f_\vee(a, b) = \max(a, b)$, $f_\vee(a, b) = 1 + a - a \cdot b$, etc.
- However, for most of them, combining infinitely many degrees leads to a useless value 1.

8. Fuzzy Marginal (cont-d)

- The only "or"-operation that makes sense is max, so

$$\mu_i(x_i) = \max_{x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n} \mu(x_1,\ldots,x_{i-1},x_i,x_{i+1},\ldots,x_n).$$

- Let's assume that $\mu_i(x_i)$ is a normal membership function: continuous and attains values 0 and 1.
- In this case, for every $r \in [0,1]$, there exists a v_i for which $\mu_i(v_i) = r$.
- How do fuzzy marginals change under re-scaling

$$\mu'(x_1,x_2,\ldots,x_n) = f(\mu(x_1,x_2,\ldots,x_n))?$$

- One can prove that they change the same way:

$$\mu'_i(x_i) = f(\mu_i(x_i)).$$

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9. How to Get a Scaling-Invariant Description of Dependence

- We want to find a description of the dependence that does not change if we re-scale all the degrees:

$$\mu(x_1, x_2, \ldots, x_n) \to \mu'(x_1, x_2, \ldots, x_n) = f(\mu(x_1, x_2, \ldots, x_n)).$$

- For $r = \mu(x_1, \ldots, x_n)$, there exists a number $r_i(x_1, \ldots, x_n)$ for which

$$\mu(x_1, \ldots, x_n) = \mu_i(r_i(x_1, \ldots, x_n)).$$

- The function $r_i(x_1, \ldots, x_n)$ describes the dependence in the sense that:
  - if we know this function and the marginals $\mu_i(x_i)$,
  - then we can reconstruct the original membership function $\mu(x_1, \ldots, x_n)$.
- One can prove that each function $r_i(x_1, \ldots, x_n)$ is scale-invariant; so, it is the desired dependence.

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10. Examples of $\mu(x_1,\ldots,x_n)=\mu_i(r_i(x_1,\ldots,x_n))$

- For the Gaussian membership function $\mu(x_1,x_2) = \exp(-x_1^2-x_2^2)$, marginal is $\mu_1(x_1)=\exp(-x_1^2)$.
- The definition of $r_1(x_1,x_2)$ becomes $\exp(-x_1^2-x_2^2) = \exp(-(r_1(x_1,x_2))^2)$.
- By taking $-\ln$ of both sides, we get $x_1^2+x_2^2 = (r_1(x_1,x_2))^2$, so $r_1(x_1,x_2)=\sqrt{x_1^2+x_2^2}$.
- For $\mu(x_1,x_2)=\dfrac{1}{1+x_1^2+x_2^2}$, we have $\mu_1(x_1)=\dfrac{1}{1+x_1^2}$.
- Here, $\dfrac{1}{1+x_1^2+x_2^2}=\dfrac{1}{1+r_1(x_1,x_2)^2}$, so $1+r(x_1,x_2)^2 = 1+x_1^2+x_2^2$, and $r_1(x_1,x_2)=\sqrt{x_1^2+x_2^2}$.
- For $\mu_1(x_1,x_2)=\exp(-|x_1|-|x_2|)$, we have $\mu_1(x_1) = \exp(-|x_1|)$, so $\exp(-|x_1|-|x_2|)=\exp(-r_1(x_1,x_2))$ and
$$r_1(x_1,x_2)=|x_1|+|x_2|.$$

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11. Examples (cont-d)

- For $\mu(x_1, x_2) = \exp(-x_1^2 - x_1 \cdot x_2 - x_2^2)$, the maximum w.r.t. x_2 is when derivative is 0, i.e., when

$$x_1 + 2x_2 = 0 \text{ and } x_2 = -\frac{x_1}{2}, \text{ so}$$

$$\mu_1(x_1) = \mu(x_1, x_2(x_1)) = \exp\left(-\frac{3}{4} \cdot x_1^2\right).$$

- Thus, $\mu(x_1, x_2) = \mu_1(r_1(x_1, r_2))$ implies

$$\exp(-x_1^2 - x_1 \cdot x_2 - x_2^2) = \exp\left(-\frac{3}{4} \cdot (r_1(x_1, x_2))^2\right).$$

- So, $(r_1(x_1, x_2))^2 = \dfrac{4}{3} \cdot (x_1^2 + x_1 \cdot x_2 + x_2^2)$ and

$$r_1(x_1, x_2) = \frac{2 \cdot \sqrt{3}}{3} \cdot \sqrt{x_1^2 + x_1 \cdot x_2 + x_2^2}.$$

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12. Auxiliary Question: What Is the Relation Between Different $r_i(x_1, \ldots, x_n)$?

- To describe the dependence, we can use $r_i(x_1, \ldots, x_n)$ for any i.
- If we know $r_i(x_1, \ldots, x_n)$, can we reconstruct $r_j(x_1, \ldots, x_n)$ for $j \neq i$?
- Let us assume that all $\mu_i(x_i)$ increase when $x_i \leq c_i$ and decrease after that.
- As values $r_i(x_1, \ldots, x_n)$, we select values $r_i \geq c_i$.
- Then, we have $r_j(x_1, \ldots, x_n) = s_{ij}^{-1}(r_i(x_1, \ldots, x_n))$, where:

$$s_{ij}(x_j) \stackrel{\text{def}}{=} \min_{x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n} r_i(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n).$$

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14. Proof That $r_i(x_1, \ldots, x_n)$ Is Scaling-Invariant

- We define $r_i(x_1, \ldots, x_n)$ by the formula

$$\mu(x_1, \ldots, x_n) = \mu_i(r_i(x_1, \ldots, x_n)).$$

- Let's re-scale membership values, i.e., replace:
    - $\mu(x_1, \ldots, x_n) \to \mu'(x_1, \ldots, x_n) = f(\mu(x_1, \ldots, x_n))$ and
    - $\mu_i(x_i) \to \mu'_i(x_i) = f(\mu_i(x_i))$.
- By applying $f(x)$ to both sides of the above equality, we get $f(\mu(x_1, \ldots, x_n)) = f(\mu_i(r_i(x_1, \ldots, x_n)))$.
- Thus, $\mu'(x_1, \ldots, x_n) = \mu'_i(r_i(x_1, \ldots, x_n))$.
- So, for the re-scaled membership degrees, we have the exact same function $r_i(x_1, \ldots, x_n)$.
- Thus, $r_i(x_1, \ldots, x_n)$ is indeed scaling-invariant.

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15. Proof of the $r_i \to r_j$ Formula

- $\mu(x_1, \ldots, x_n) = \mu_i(r_i(x_1, \ldots, x_n))$ implies that

$$\mu_j(x_j) = \max_{x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_n} \mu(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n) =$$

$$\max_{x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_n} \mu_i(r_i(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n)).$$

- $\mu_i(x_i)$ is strictly decreasing for $x_i \geq c_i$.
- Thus, $\mu_i(r_i(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n))$ is max $\Leftrightarrow$ $r_i(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n)$ is min; so, $\mu_j(x_j) =$

$$\max_{x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_n} \mu_i(r_i(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n)).$$

- Thus, $\mu_j(x_j) = \mu_i(s_{ij}(x_j))$ and $\mu_i(x_i) = \mu_j(s_{ij}^{-1}(x_i))$.
- From $\mu(x_1, \ldots, x_n) = \mu_i(r_i(x_1, \ldots, x_n))$, we get

$$\mu(x_1, \ldots, x_n) = \mu_j(s_{ij}^{-1}(r_i(x_1, \ldots, x_n)).$$

- Thus, $\mu(x_1, \ldots, x_n) = \mu_j(r_j(x_1, \ldots, x_n))$ for $r_j(x_1, \ldots, x_n) = s_{ij}^{-1}(r_i(x_1, \ldots, x_n))$. Q.E.D.

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