Isn't Every Sufficiently Complex Logic Multi-Valued Already: Lindenbaum-Tarski Algebra and Fuzzy Logic Are Both Particular Cases of the Same Idea

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1. A Gap Between Fuzzy Logic and the Traditional 2-Valued Fuzzy Logic

- One of the main ideas behind fuzzy logic is that:
  - in contrast to the traditional 2-valued logic, in which every statement is either true or false,
  - in fuzzy logic, we allow intermediate degrees.
- In other words, fuzzy logic is an example of a *multi-valued* logic.
- This led to a misunderstanding between researchers in fuzzy and traditional logics.
- Fuzzy logic books claim that the 2-valued logic cannot describe intermediate degrees.
- On the other hand, 2-valued logicians criticize fuzzy logic for using "weird" intermediate degrees.

2. What We Do in This Paper

- We show that the mutual criticism is largely based on a misunderstanding.
- It *is* possible to describe intermediate degrees in the traditional 2-valued logic.
- However, such a representation is complicated.
- The main advantage of fuzzy techniques is that they provide a simply way of doing this.
- And simplicity is important for applications.
- We also show that the main ideas of fuzzy logic are consistent with the 2-valued foundations.
- Moreover, they naturally appear in these foundations if we try to adequately describe expert knowledge.
- We hope to help researchers from both communities to better understand each other.

3. Source of Multi-Valuedness in Traditional Logic: Gödel's Theorem

- A naive understanding of the 2-valued logic assumes that every statement S is either true or false.
- This is possible in simple situations.
- However, Gödel's showed that this not possible for complex theories.
- Gödel analyzed arithmetic – statements obtained
  - from basic equalities and inequalities between polynomial expressions
  - by propositional connectives &, $\vee$, $\neg$, and quantifiers over natural numbers.
- He showed that it is not possible to have a theory T in which for every statement S, either $T \models S$ or $T \models \neg S$.

4. We Have, in Effect, at Least Three Different Truth Values

- Due to Gödel's theorem, there exist statements S for which $T \not\models S$ and $T \not\models \neg S$. So:
  - while, legally speaking, the corresponding logic is 2-valued,
  - in reality, such a statement S is neither true nor false.
- Thus, we have more than 2 possible truth values.
- At first glance, we have 3 truth values: "true", "false", and "unknown".
- However, different "unknown" statements are not necessarily provably equivalent to each other.
- So, we may have more than 3 truth values.

5. How Many Truth Values Do We Actually Have

- It is reasonable to consider the following equivalence relation between statements A and B:

$$\models (A \Leftrightarrow B)$$

- Equivalence classes with respect to this relation can be viewed as the actual truth values.
- The set of all such equivalence classes is known as the *Lindenbaum-Tarski algebra*.
- Lindenbaum-Tarski algebra shows that any sufficiently complex logic is, in effect, multi-valued.
- However, this multi-valuedness is different from the multi-valuedness of fuzzy logic.
- We show that there is another close-to-fuzzy aspect of multi-valuedness of the traditional logic.

6. Need to Consider Several Theories

- In the previous section, we considered the case when we have a single theory T.
- Gödel's theorem states that:
  - for every given theory T that includes formal arithmetic,
  - there is a statement S that can neither be proven nor disproven in this theory.
- This statement S can neither be proven not disproven based on the axioms of theory T.
- So, a natural idea is to consider additional reasonable axioms that we can add to T.

7. Need to Consider Several Theories (cont-d)

- This is what happened in geometry with the V-th postulate P – that
  - for every line ℓ in a plane and for every point P outside this line,
  - there exists only one line ℓ' which passes through P and is parallel to ℓ.
- It turned out that neither P not $\neg P$ can be derived from all other (more intuitive) axioms of geometry.
- So, a natural solution is to explicitly add this statement as a new axiom.
- If we add its negation, we get Lobachevsky geometry – historically the first non-Euclidean geometry.

8. Need to Consider Several Theories (cont-d)

- A similar thing happened in set theory, with the Axiom of Choice and Continuum Hypothesis.
- They cannot be derived or rejected based on the other (more intuitive) axioms of set theory.
- Thus, they (or their negations) have to be explicitly added to the original theory.
- The new – extended – theory covers more statements that the original theory T.
- However, the same Gödel's theory still applies to the new theory:
  - there are statements that
  - can neither be deduced nor rejected based on this new theory.
- Thus, we need to add one more axiom, etc.

9. We Have a Family of Theories

- So, instead of a *single* theory, it makes sense to consider a *family* of theories $\{T_\alpha\}_\alpha$.
- In the above description, we end up with a family which is *linearly ordered* in the sense that:
  - for every two theories T_α and T_β,
  - either $T_\alpha \models T_\beta$ or $T_\beta \models T_\alpha$.
- However, it is possible that on some stage, different groups of researchers select two different axioms.
- In this case, we will have two theories which are not derivable from each other.
- Thus, we have a family of theories which is not linearly ordered.

10. How's This Applicable to Expert Knowledge?

- We can select only the statements in which experts are 100% sure, and we get one possible theory.
- We can add statements S for which the expert's degree of confidence $d(S)$ exceeds a certain threshold α:

$$\{S : d(S) \geq \alpha\}.$$

- For different α, we get different theories T_α.
- For example, if we select $\alpha = 0.7$, then:
  - For every x for which $\mu_{\text{small}}(x) \geq 0.7$, we consider $S(x)$ ("x is small") to be true.
  - For all other objects x, we consider $S(x)$ to be false.
- Similarly, we only keep "if-then" rules for which the expert's degree of confidence in these rules is ≥ 0.7.

11. Once We Have a Family of Theories, How Can We Describe the Truth of a Statement?

- If we have a single theory T, then for every S:
  - either $T \models S$, i.e., the statement S is true in the theory T,
  - or $T \not\models \neg S$, i.e., S is not true in the theory T.
- In general:
  - to describe whether a statement S is true or not,
  - we should consider the values corresponding to all the theories T_α.
- So, we should consider the whole set

$$\deg(S) \stackrel{\text{def}}{=} \{\alpha : T_\alpha \models S\}.$$

- This set is our degree of belief that S is true – i.e., in effect, the truth value of the statement S.

12. Logical Operations on the New Truth Values

- If a theory T_α implies both S and S', then this theory implies their conjunction $S \,\&\, S'$ as well.
- So, the truth value of the conjunction includes the intersection of truth value sets corresponding to S and S':

$$\deg(S \,\&\, S') \supseteq \deg(S) \cap \deg(S').$$

- Similarly, if a theory T_α implies either S or S', then this theory also implies the disjunction $S \vee S'$.
- Thus, the truth value of the disjunction includes the union of truth value sets corresponding to S and S':

$$\deg(S \vee S') \supseteq \deg(S) \cup \deg(S').$$

13. What Happens in the Simplest Case, When the Theories Are Linearly Ordered?

- If the theories T_α are linearly ordered, then, once $T_\alpha \models S$ and $T_\beta \models T_\alpha$, we also have $T_\beta \models S$.
- Thus, with every T_α, the truth value $\deg(S) = \{\alpha : T_\alpha \models S\}$ includes:
  - with each index α,
  - the indices of all the stronger theories – i.e., all the theories T_β for which $T_\beta \models T_\alpha$.
- In particular, for a finite family of theories, each degree is equal to $D_{\alpha_0} \stackrel{\text{def}}{=} \{\alpha : T_\alpha \models T_{\alpha_0}\}$ for some α_0.
- In terms of the linear order $\alpha \le \beta \Leftrightarrow T_\alpha \models T_\beta$, this degree takes the form $D_{\alpha_0} = \{\alpha : \alpha \le \alpha_0\}$.
- We can thus view α_0 as the degree of truth of the statement S: $\mathrm{Deg}(S) \stackrel{\text{def}}{=} \alpha_0$.

14. Linearly Ordered Case (cont-d)

- In case of expert knowledge, this means that we consider the smallest degree of confidence d for which:
  - we can derive the statement S
  - if we allow all the expert's statements whose degree of confidence is at least d.
- These sets D_α are also linearly ordered: one can easily show that $D_\alpha \subseteq D_\beta \Leftrightarrow \alpha \le \beta$.
- The intersection of sets D_α and D_β simply means that we consider the set $D_{\min(\alpha,\beta)}$.
- The union of sets D_α and D_β simply means that we consider the set $D_{\max(\alpha,\beta)}$.
- Thus, the statements about & and $\vee$ take the form:

$$\text{Deg}(S \,\&\, S') \ge \min(\text{Deg}(S), \text{Deg}(S'));$$
$$\text{Deg}(S \vee S') \ge \max(\text{Deg}(S), \text{Deg}(S')).$$

15. Relation to Fuzzy

- We have shown that:

$$\mathrm{Deg}(S \mathbin{\&} S') \ge \min(\mathrm{Deg}(S), \mathrm{Deg}(S'));$$

$$\mathrm{Deg}(S \vee S') \ge \max(\mathrm{Deg}(S), \mathrm{Deg}(S')).$$

- The above formulas are very similar to the formulas of the fuzzy logic corresponding to min and max.
- The only difference is that we get $\ge$ instead of $=$.
- Thus, fuzzy logic ideas can be indeed naturally obtained in the classical 2-valued environment.
- They can be interpreted as a particular case of the same general idea as the Lindenbaum-Tarski algebra.

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