How to Fuse Expert Knowledge: Not Always "And" but a Fuzzy Combination of "And" and "Or"

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Need to Fuse . . .
Examples
Fusing Expert . . .
Example
Need to Consider the . . .
In General, How . . .
How to Define Degree . . .
Resulting Definition of . . .
Discussion

1. Need to Fuse Knowledge of Different Experts

- Expert estimates of different quantities are usually not very accurate.
- In situations when measurements are possible, they are more accurate than expert estimates.
- When we can perform measurements:
  - we can further increase the measurement accuracy
  - if we use several different measuring instruments and then combine ("fuse") their results.
- It is known that such combinations are usually more accurate than all original measurement results.

Need to Fuse . . .
Examples
Fusing Expert . . .
Example
Need to Consider the . . .
In General, How . . .
How to Define Degree . . .
Resulting Definition of . . .
Discussion

2. Need to Fuse Knowledge (cont-d)

- In many situations, measurements are not realistically possible, so we have to rely on expert estimates only.
- In such situations:
  - we can increase the accuracy of the resulting estimates the same way:
  - by combining (fusing) estimates of several experts.

Need to Fuse . . .
Examples
Fusing Expert . . .
Example
Need to Consider the . . .
In General, How . . .
How to Define Degree . . .
Resulting Definition of . . .
Discussion

3. Examples

- To estimate the temperature, we can ask two experts.
- Suppose that:
  - one expert states that the temperature is between 22 and 25 degree C, and
  - another expert states the temperature is in the low seventies, i.e., between 70 and 75 F;
  - this corresponds to between 21 and 24 C.
- Then we can conclude that the actual temperature is larger than 22 C and smaller than 24 C – i.e., the actual temperature is between 22 and 24 C.
- If we only ask one expert, we get an interval of width 3 that contains the actual temperature.
- But by fusing the opinions of the two experts, we get a narrower interval $[22, 24]$ of width 2.

Need to Fuse ...
Examples
Fusing Expert ...
Example
Need to Consider the ...
In General, How ...
How to Define Degree ...
Resulting Definition of ...
Discussion

4. Examples (cont-d)

- So, we have indeed increased the accuracy.
- Fusion is also possible on a non-quantitative level.
- For example, we can ask experts whether the wind is weak, moderate, or strong.
- Suppose that:
  - one expert says that the wind is not weak, while
  - another expert says that the wind is not strong.
- By combining the opinions of both experts, we can conclude that the wind is moderate.
- On the other hand, if we only to one of the experts, we would not be able to come to this conclusion.

Need to Fuse . . .
Examples
Fusing Expert . . .
Example
Need to Consider the . . .
In General, How . . .
How to Define Degree . . .
Resulting Definition of . . .
Discussion

5. Fusing Expert Knowledge: Non-Fuzzy Case

- Let us start with the case when expert estimates are crisp (non-fuzzy).
- So, for each possible value of the estimated quantity, the expert is:
  - either absolutely sure that this value is possible
  - or is absolutely sure that the given value is not possible.
- In this case, each expert estimate provides us with a set of possible values of the corresponding quantity.
- In most practical cases, this set is an interval $[\underline{x}, \overline{x}]$.

Need to Fuse . . .
Examples
Fusing Expert . . .
Example
Need to Consider the . . .
In General, How . . .
How to Define Degree . . .
Resulting Definition of . . .
Discussion

6. Non-Fuzzy Case (cont-d)

- In these terms, when we have estimates of two different experts, this means that:
  - based on the opinions of the first expert, we form a set S_1 of possible values;
  - also, based on the opinions of the second expert, we form a set S_2 of possible values.
- In general, different experts take into account different aspects of the situation.
- For example, the first expert may know the upper bound $\overline{x}$ on the corresponding quantity.
- In this case, the set S_1 consists of all the numbers which are smaller than or equal to $\overline{x}$, i.e., $S_1 = (-\infty, \overline{x}]$.
- The second expert may know the lower bound $\underline{x}$, in which case $S_2 = [\underline{x}, \infty)$.

Need to Fuse . . .
Examples
Fusing Expert . . .
Example
Need to Consider the . . .
In General, How . . .
How to Define Degree . . .
Resulting Definition of . . .
Discussion

7. Non-Fuzzy Case (cont-d)

- A natural way to fuse the knowledge is to consider numbers which are possible according to both experts.
- In mathematical terms, we consider the intersection $S_1 \cap S_2$ of the two sets S_1 and S_2.
- A problem occurs when this intersection is empty, i.e., when the opinions of two experts are inconsistent.
- This happens: experts are human and can thus make mistakes.
- In this case, an extreme option is to say that:
  - since experts are not consistent with each other,
  - this means that we do not trust what each of them says,
  - so we can as well ignore both opinions; the result of fusion is then the whole real line.

Need to Fuse . . .
Examples
Fusing Expert . . .
Example
Need to Consider the . . .
In General, How . . .
How to Define Degree . . .
Resulting Definition of . . .
Discussion

8. Non-Fuzzy Case (cont-d)

- A more reasonable option is:
  - to conclude that, yes, both experts cannot be true, but
  - we cannot conclude that both are wrong.
- They are experts after all, so it is reasonable to assume that one of them is right.
- In this case, the result of the fusion is the union $S_1 \cup S_2$ of the two sets.
- In other words, the fusion $S_1 \, f \, S_2$ of the sets S_1 and S_2 has the following form;
  - if $S_1 \cap S_2 \neq \emptyset$, then $S_1 \, f \, S_2 = S_1 \cap S_2$;
  - otherwise, if $S_1 \cap S_2 = \emptyset$, then $S_1 \, f \, S_2 = S_1 \cup S_2$.

Need to Fuse . . .
Examples
Fusing Expert . . .
Example
Need to Consider the . . .
In General, How . . .
How to Define Degree . . .
Resulting Definition of . . .
Discussion

9. Example

- Suppose that:
  - one expert says that the temperature is between 22 and 25, and
  - another one claims that it is between 18 and 21.
- In this case, the intersection of the corresponding intervals $[22, 25]$ and $[18, 21]$ is empty.
- This means that the experts cannot be both right.
- What we can conclude:
  - if we still believe that one of them is right
  - is that the temperature is either between 22 and 25 or between 18 and 21.

Need to Fuse . . .
Examples
Fusing Expert . . .
Example
Need to Consider the . . .
In General, How . . .
How to Define Degree . . .
Resulting Definition of . . .
Discussion

10. Need to Consider the Fuzzy Case

- In practice, experts are rarely absolutely confident about their opinions.
- Usually, they are only confident to a certain degree.
- As a result, to adequately describe expert knowledge, we need to describe:
  - for each number x,
  - the degree to which, according to this expert, the number x is possible.
- This is the *fuzzy logic* approach; in the computer:
  - "true" (= "absolutely certain") is usually represented as 1, and
  - "false" (= "absolutely certain this is false") is represented as 0.

Need to Fuse . . .
Examples
Fusing Expert . . .
Example
Need to Consider the . . .
In General, How . . .
How to Define Degree . . .
Resulting Definition of . . .
Discussion

11. Need to Consider the Fuzzy Case (cont-d)

- It is therefore reasonable to describe intermediate degrees of confidence by numbers between 0 and 1.
- Thus, to describe an expert's estimate, we need to have a function $\mu(x)$ that assigns:
  - to each value x of the corresponding quantity,
  - a number $\mu(x) \in [0, 1]$ that describes to what extent the value x is possible.
- Such a function is known as a *membership function*, or, alternatively, a *fuzzy set*.
- From this viewpoint, to be able to fuse expert estimates, we need to be able to fuse fuzzy sets.
- A traditional approach to fusing fuzzy knowledge simply takes the intersection.

Need to Fuse . . .
Examples
Fusing Expert . . .
Example
Need to Consider the . . .
In General, How . . .
How to Define Degree . . .
Resulting Definition of . . .
Discussion

12. Need to Consider the Fuzzy Case (cont-d)

- The intersection is usually normalized, i.e., multiplied by a constant so that the maximum value is 1.
- However, this does not work if the expert opinions are inconsistent.
- We should therefore take into account that the expert opinions can be inconsistent.
- Another option is that they are consistent to a certain degree.
- How can take this into account?

Need to Fuse . . .
Examples
Fusing Expert . . .
Example
Need to Consider the . . .
In General, How . . .
How to Define Degree . . .
Resulting Definition of . . .
Discussion

13. In General, How Notions Are Generalized to the Fuzzy Case

- The usual way to generalize different notions to the fuzzy case is as follows.
- First, we describe the original notion in logical terms, by using “and”, “or”, and quantifiers:
  - “for all” (which is, in effect, infinite “and”) and
  - “exists” (which is, in effect, infinite “or”).
- Then, we replace each “and” operation with the fuzzy “and”-operation $f_\&(a, b)$ (also known as t-norm).
- We replace every “or”-operation with the fuzzy “or”-operation $f_\vee(a, b)$ (also known as t-conorm).
- In selecting the t-norms and t-conorms, we need to be careful, in the following sense.

Need to Fuse . . .
Examples
Fusing Expert . . .
Example
Need to Consider the . . .
In General, How . . .
How to Define Degree . . .
Resulting Definition of . . .
Discussion

14. How Notions Are Generalized (cont-d)

- If we have a universal quantifier – i.e., an infinite "and", and
  - we use, e.g., a product t-norm $f_\&(a, b) = a \cdot b$,
  - then the product of infinitely many values smaller than 0 will be most probably simply 0.
- So, if we have a universal quantifier, the only reasinable t-norm is $f_\&(a, b) = \min(a, b)$.
- Similarly, if we have an existential quantifier – i.e., an infinite "or" – and
  - we use, e.g., $f_\vee(a, b) = a + b - a \cdot b$,
  - then the result of applying this operation to infinitely many values > 0 will be 1.
- So, if we have an existential quantifier, the only reasonable t-conorm is $f_\vee(a, b) = \max(a, b)$.

Need to Fuse . . .
Examples
Fusing Expert . . .
Example
Need to Consider the . . .
In General, How . . .
How to Define Degree . . .
Resulting Definition of . . .
Discussion

15. How to Define Degree of Consistency

- Let us use the above-described general approach to define the degree of consistency.
- In the non-fuzzy case, two expert opinions are consistent if:
  - there exists a value x for which
  - both the first expert and the second expert agree that x is possible.
- For each real number x representing a possible value of the quantity of interest:
  - let $\mu_1(x)$ denote the degree to which the first expert believes the value x to be possible, and
  - let $\mu_2(x)$ denote the degree to which the second experts believes the value x to be possible.

Need to Fuse . . .
Examples
Fusing Expert . . .
Example
Need to Consider the . . .
In General, How . . .
How to Define Degree . . .
Resulting Definition of . . .
Discussion

16. Degree of Consistency (cont-d)

- Then, for each value x, the degree to which both experts consider the value x to be possible is

$$f_{\&}(\mu_1(x), \mu_2(x)).$$

- In line with the above general scheme for generalizing notions into fuzzy:
  - the existential quantifier over x
  - is translated into maximum over x
  - which corresponds to the use of the maximum "or"-operation $f_{\vee}(a, b) = \max(a, b)$.
- Thus, we get the following formula for the degree $d(\mu_1, \mu_2)$ for which two membership functions are consistent:

$$d(\mu_1, \mu_2) = \max_x f_{\&}(\mu_1(x), \mu_2(x)).$$

Need to Fuse ...
Examples
Fusing Expert ...
Example
Need to Consider the ...
In General, How ...
How to Define Degree ...
Resulting Definition of ...
Discussion

17. Degree of Consistency (cont-d)

- Accordingly:
  - in line with a general description of negation in fuzzy logic,
  - the degree to which the expert opinions are *in*consistent can be computed as $1 - d(\mu_1, \mu_2)$.

Need to Fuse . . .
Examples
Fusing Expert . . .
Example
Need to Consider the . . .
In General, How . . .
How to Define Degree . . .
Resulting Definition of . . .
Discussion

18. Resulting Definition of Fusion

- As we mentioned, in the non-fuzzy case, the value x belongs to the fused set if:
  - either the two sets describing expert opinions are consistent, and x belongs to their intersection,
  - or the two sets describing expert opinions are inconsistent, and x belongs to their union.
- Let us use the general methodology to generalize the above description to the fuzzy case.
- For each x:
  - we know the degree $d(\mu_1, \mu_2)$ to which the experts are consistent, and
  - we know the degree $f_\&(\mu_1(x), \mu_2(x))$ to which x belongs to the intersection.

Need to Fuse . . .
Examples
Fusing Expert . . .
Example
Need to Consider the . . .
In General, How . . .
How to Define Degree . . .
Resulting Definition of . . .
Discussion

19. Resulting Definition of Fusion (cont-d)

- Thus, the degree to which the expert opinions are consistent *and* x belongs to the intersection is

$$f_{\&}(d(\mu_1,\mu_2), f_{\&}(\mu_1(x),\mu_2(x))) = f_{\&}(d(\mu_1,\mu_2),\mu_1(x),\mu_2(x)).$$

- Similarly, for each x:
  - we know the degree $1-d(\mu_1,\mu_2)$ to which the experts are inconsistent, and
  - we know the degree $f_{\vee}(\mu_1(x),\mu_2(x)) = \max(\mu_1(x),\mu_2(x))$ to which x belongs to the union.
- Thus, the degree to which the expert opinions are inconsistent *and* x belongs to the union is

$$f_{\&}(1-d(\mu_1,\mu_2), \max(\mu_1(x),\mu_2(x))).$$

Need to Fuse . . .
Examples
Fusing Expert . . .
Example
Need to Consider the . . .
In General, How . . .
How to Define Degree . . .
Resulting Definition of . . .
Discussion

20. Resulting Definition of Fusion (cont-d)

- To find the degree $\mu(x)$ to which the value x belongs to the fused set, we need to apply the "or"-operation:

$$\mu(x) = \max(d_1(x), d_2(x)), \text{ where}$$

$$d_1(x) \stackrel{\text{def}}{=} f_{\&}(d(\mu_1, \mu_2), \mu_1(x), \mu_2(x)),$$

$$d_2(x) \stackrel{\text{def}}{=} f_{\&}(1 - d(\mu_1, \mu_2), \max(\mu_1(x), \mu_2(x)))), \text{ and}$$

$$d(\mu_1, \mu_2) = \max_x f_{\&}(\mu_1(x), \mu_2(x)).$$

Need to Fuse . . .
Examples
Fusing Expert . . .
Example
Need to Consider the . . .
In General, How . . .
How to Define Degree . . .
Resulting Definition of . . .
Discussion

21. Discussion

- We can see that this fused fuzzy set is not exactly "and", it is not exactly "or".
- It is a fuzzy combination of "and" and "or".
- When $f_{\&}(a, b) = \min(a, b)$, our formula for $\mu(x)$ can be simplified into the following:

$$\max(\min(\mu_1(x), \mu_2(x)), \min(1-d(\mu_1, \mu_2), \max(\mu_1(x), \mu_2(x)))).$$

Need to Fuse . . .
Examples
Fusing Expert . . .
Example
Need to Consider the . . .
In General, How . . .
How to Define Degree . . .
Resulting Definition of . . .
Discussion

22. Example

- Let us consider a simple case when:
  - the "and"-operation is minimum, and
  - the membership functions are triangular functions of the same width.
- To make the computations even easier, let us select, as a starting point for measuring x:
  - the arithmetic average
  - between the most probable values corresponding to the two experts.
- Let us select the measuring unit so that the half-width of each membership function is 1.
- In this case, the triangular membership functions are described, for soem $a > 0$, by the formulas

  $\mu_1(x) = \max(0, 1-|x-a|)$ and $\mu_2(x) = \max(0, 1-|x+a|)$.

Need to Fuse . . .
Examples
Fusing Expert . . .
Example
Need to Consider the . . .
In General, How . . .
How to Define Degree . . .
Resulting Definition of . . .
Discussion

23. Example (cont-d)

- This value a is the half of the difference between:
  - the most probable value (a) according to the first expert and
  - the most probable value according to the second expert ($-a$):

$$a = \frac{a - (-a)}{2}.$$

- When $a \geq 1$, the two membership functions have no intersection at all, so $d(\mu_1, \mu_2) = 0$.
- Then, the fused set is simply their union $\max(\mu_1(x), \mu_2(x))$.
- It is a bi-modal set whose graph consists of the two original triangles.
- The more interesting case is when $a < 1$.
- In this case, the two sets have some degree of intersection.

24. Example (cont-d)

- For such values a, the intersection $f_\&(\mu_1(x), \mu_2(x))$ is also a triangular function $\max(0, 1 - a - |x|)$.
- The maximum $d(\mu_1, \mu_2)$ of this function is attained when $x = 0$ and is equal to $1 - a$.
- Correspondingly, the degree to which the two expert opinions are inconsistent is equal to

$$1 - d(\mu_1, \mu_2) = 1 - (1 - a) = a.$$

- By applying our formula, we can now conclude the following.
- When $a \le 0.5$, the fused expression is still a *fuzzy number*, i.e., $\mu(x)$ first increases and then decreases.

Need to Fuse . . .
Examples
Fusing Expert . . .
Example
Need to Consider the . . .
In General, How . . .
How to Define Degree . . .
Resulting Definition of . . .
Discussion

25. Example (cont-d)

- $\mu(x)$ starts being non-zero at $x = -1 - a$.
- Between the value $-1 - a$ and -1, it grows as $\mu(x) = x - (-1 - a) = 1 + a - x$.
- Between the values $x = -1$ and $x = -(1 - 2a)$, the fused function remains constant $\mu(x) = a$.
- Between $x = -(1 - 2a)$ and $x = 0$, it grows as $\mu(x) = 1 - a + x$, until it reaches the value $1 - a$.
- Then, for x from 0 to $1 - 2a$, it decreases as $\mu(x) = 1 - a - x$ until it reaches the value a for $x = 1 - 2a$.
- Then, the value stays constant $\mu(x) = a$ until we reach $x = 1$.
- Finally, for x between 1 and $1 + a$, the values decreases as $\mu(x) = (1 + a) - x$.
- It reaches 0 for $x = 1 + a$ – and stays 0 after that.

Need to Fuse . . .
Examples
Fusing Expert . . .
Example
Need to Consider the . . .
In General, How . . .
How to Define Degree . . .
Resulting Definition of . . .
Discussion

26. Example (cont-d)

- We can normalize the resulting function, by dividing it by its largest possible value $1 - a$.
- Then, the constant levels increase to $\dfrac{a}{1-a}$.
- When $a > 0.5$, we simply get the union cut-off at level $1 - a$, i.e., $\mu(x) = \min(1 - a, \max(\mu_1(x), \mu_2(x)))$.

Need to Fuse . . .
Examples
Fusing Expert . . .
Example
Need to Consider the . . .
In General, How . . .
How to Define Degree . . .
Resulting Definition of . . .
Discussion

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Need to Fuse . . .
Examples
Fusing Expert . . .
Example
Need to Consider the . . .
In General, How . . .
How to Define Degree . . .
Resulting Definition of . . .
Discussion