Logarithms Are Not Infinity: A Rational Physics-Related Explanation of the Mysterious Statement by Lev Landau

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1. Physicists Use Intuition

- Physicists have been very successful in predicting physical phenomena.
- Many fundamental physical phenomena can be predicted with very high accuracy.
- The question is: how do physicists come up with the corresponding models?
- In this, physicists often use their intuition.
- This intuition is, however, difficult to learn, because it is not formulated in precise terms.
- It is imprecise, it is intuition, after all.

2. Can We Formalize Physicists' Intuition – at Least Some of It?

- It would be great to be able to emulate at least some of this intuition in a computer-based systems.
- Then, the same successful line of reasoning can be used to solve many other problems.
- Computers, however, only understand precise terms.
- So, to be able to emulate physicists' intuition on a computer, we need describe it in precise terms.

3. An Example of Physicists' Intuition: Landau's Statement about Logarithms

- Nobel-prize physicist Lev Landau often said that "logarithms are not infinity".
- This means that, in some sense, the logarithm of an infinite value is not really infinite.
- From the purely mathematical viewpoint, this statement by Landau makes no sense.
- Of course, the limit of $\ln(x)$ when x tends to infinity is infinite.
- This was a statement actively used by a Nobel-prize winning physicist.
- So, we cannot just ignore it as a mathematically ignorant nonsense.

4. Why Are Infinities Important in the First Place?

- In physics, everything is finite, infinities are mathematical abstractions, what is the big deal?
- Let us compute the overall mass m of an electron.
- According to special relativity theory, $m = E/c^2$, where $E = m_0 \cdot c^2 + E_{\rm el}$.
- Here $E_{\rm el}$ is the energy of the electron's electric field.
- According to the same relativity theory, the speed of all communications is limited by the speed of light.
- As a result, any elementary particle must be pointwise.
- Otherwise, different parts – due to speed-of-light bound – would constitute different sub-particles.
- The electric field $\vec{E}$ is $\vec{E}(x) = c_1 \cdot \dfrac{q}{r^2}$.

5. Why Are Infinities Important (cont-d)

- The field's energy density $\rho(x)$ is proportional to the square of the field: $\rho(x) = c_2 \cdot (\vec{E}(x))^2$.
- So, $\rho(x) = c_3 \cdot \dfrac{1}{r^4}$, where $c_3 \stackrel{\text{def}}{=} c_2 \cdot (c_1 \cdot q)^2$.
- Thus, the overall energy of the electric field can be found if we integrate this density over the whole space:

$$E_{\text{el}} = \int \rho(x)\,dx = c_3 \cdot \int \frac{1}{r^4}\,dx = c_3 \cdot \int_0^\infty \frac{4\pi \cdot r^2}{r^4}\,dr =$$

$$c_4 \cdot \int \frac{1}{r^2}\,dr = -c_4 \cdot \left.\frac{1}{r}\right|_0^\infty .$$

- For $r = 0$, we get a physically meaningless infinity!

6. This Infinity Problem is Ubiquitous

- The problem is not just in the specific formulas for the Coulomb law, the problem is much deeper.
- Many interactions are *scale-invariant* in the sense that they have no physically preferable unit of length.
- If we change the unit of length to a new one which is λ times smaller, then we get $r' = \lambda \cdot r$.
- Scale-invariance means that all the physical equations remain the same after this change.
- Of course, we need to appropriately change the unit for measuring energy density, to $\rho \to \rho' = c(\lambda) \cdot \rho$.
- Suppose that in the original units, we have $\rho(r) = f(r)$ for some function f.
- Then in the new units, we will have $\rho'(r') = f(r')$ for the exact same function $f(r)$.

7. Infinity Problem is Ubiquitous (cont-d)

- Here, $\rho' = c(\lambda)\cdot\rho$ and $r' = \lambda\cdot r$, so $c(\lambda)\cdot\rho(r) = f(\lambda\cdot r)$ and $c(\lambda)\cdot f(r) = f(\lambda\cdot r)$.
- It is known that every measurable solution of this equation has the form $f(r) = c\cdot r^\alpha$ for some c and α.
- Thus, $\rho(r) = c\cdot r^\alpha$ and therefore, the overall energy of the corresponding field is equal to

$$\int \rho(x)\,dx = \int c\cdot r^\alpha\,dx = \int_0^\infty c\cdot r^\alpha\cdot 4\pi\cdot r^2\,dr = c'\cdot\int_0^\infty r^{2+\alpha}\,dr.$$

- When $\alpha \neq -3$, this integral is proportional to $r^{3+\alpha}|_0^\infty$.
- When $\alpha < -3$, this value is infinite at $r = 0$.
- When $\alpha > -3$, this value is infinite for $r = \infty$.
- In both cases, we get infinite energy.
- When $\alpha = -3$, the integral is proportional to $\ln(x)|_0^\infty$.

8. Infinity Problem is Ubiquitous (cont-d)

- Logarithm is infinite for $r = 0$ (when it is $-\infty$) and for $r = \infty$ (when it is $+\infty$), so the difference is ∞.
- The situation is not limited to our 3-dimensional proper space (corresponding to 4-dimensional space-time).
- It can be observed in space-time of any dimension d, where the area of the sphere is $\sim r^{d-1}$.
- Thus the overall energy is proportional to the integral of $r^\alpha \cdot r^{d-1} = r^{\alpha+d-1}$.
- If $\alpha \neq -d$, this integral is $\sim r^{\alpha+d}$ and infinite for $r = 0$ (when $\alpha < -d$) or for $r = \infty$ (when $\alpha > -d$).
- If $\alpha = -d$, then the integral is proportional to $\ln(x)|_0^\infty$ and is, thus, infinite as well.

9. In Reality, Infinities Are an Idealization

- In the above computations, we assumed that the distance r can take any value from 0 to infinity.
- In reality, the distance r cannot be too large: it cannot exceed the current radius R of the Universe.
- Similarly, the distance r cannot be too small.
- When $r_0 \approx 10^{-33}$ cm, quantum effects become so large that the notion of exact distance becomes impossible.
- When physicists talk about infinite values, they mean that the value is very large.
- When physicists talk about 0 values, they mean that the corresponding values are very small.
- The quantum-effects distance 10^{-33} cm is much smaller than anything we measure.
- So, we can safely take this distance to be 0.

10. What Should We Do

- The notions "very large" and "very small" are clearly imprecise.
- To properly describe these notions, it makes sense to use techniques of fuzzy logic.
- This is something we will try to do, and this is something that we encourage interested readers to try.
- While such a formalization is still not done, what can we do?

11. Since There Are No Infinities, What Is the Problem?

- Instead of $r = 0$ as the lower bound on the integral, we can use the quantum distance $r_0 = 10^{-33}$ cm.
- Then, we get a finite value proportional to $1/r_0$.
- However, r_0 is approximately 10^{-20} of the observed electron radius.
- Thus, the overall energy of the electron's electric field is 10^{20} times larger than we expected – too large.
- Similarly, in all other cases.
- If we take a very large value, and raise it to a power, we still get a very large value.

12. But With Logarithms It Is Different: a Physical Explanation of Landau's Statement

- The situation with $\ln(x)$ is drastically different.
- For $x \approx 10^{20}$, we get $\ln(10^{20}) = 20 \cdot \ln(10) \approx 46$.
- If the coefficient of proportionality is 0.01, the resulting term is smaller than 1!
- This is probably what Landau had in mind.
- For power law $y = r^\alpha$, the value of y is too large to be meaningful.
- For $y = \ln(r)$, even if r is very large, we get a very reasonable y.
- Of course, this is just a qualitative explanation.
- To get a quantitative explanation, we need to further develop fuzzy formalization of this idea.

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