How General Is Fuzzy Decision Making?

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1. Decision making in general: a brief reminder

- In many practical situations, we make decisions, i.e., we select one of the alternatives.
- Let us first consider situations in which we have full information about the alternatives.
- For example, when we buy a house, we know its location, its price, its size, its age, etc.
- The information about alternatives is usually described by numerical values of different characteristics.
- For example, for a house, we may know its price, the distance to a nearby school, the square footage, etc.

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2. Decision making in general (cont-d)

- Let us denote the number of such numerical characteristics by n, and the values of these characteristic by

$$x_1, \ldots, x_n.$$

- In these terms, each alternative can be represented by the corresponding tuple of values

$$x = (x_1, \ldots, x_n).$$

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3. Fuzzy decision making: a brief reminder

- In some cases, people have a very clear description of how they want to make a decision.
- For example, a person with a big family may be interested mostly in the square footage.
- So, this person may have:
  - a fixed amount of money a_1 that he/she is willing to pay for a house,
  - so that the price x_1 does not exceed a_1,
  - a fixed maximum distance a_2 from a school,
  - so that the actual distance x_2 is smaller than or equal to a_2, and
  - the desired square footage a_3 so that the actual square footage x_3 is larger than or equal to a_3.

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4. Fuzzy decision making (cont-d)

- Such a person is willing to buy any house $x = (x_1, x_2, x_3)$ for which $x_1 \le a_1$, $x_2 \le a_2$, and $x_3 \ge a_3$.
- However, such decision makers are rare.
- Most of the time, when people make decisions, they do not formulate their decision criteria in precise terms.
- Instead, they formulate them by using imprecise (fuzzy) words from natural language.
- A person looking for a house will probably say that he/she wants a house which is:
  - located in a good neighborhood,
  - reasonably large,
  - not too expensive,
  - not far away from the stores and entertainment district, etc.

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5. Fuzzy decision making (cont-d)

- All these terms – *good* neighborhood, *reasonably* large, etc. are imprecise.
- A natural way to describe these criteria in precise terms is to use *fuzzy techniques*.
- These techniques were designed by Lotfi Zadeh specifically for translating:
  - words from natural language
  - into precise, computer-understandable terms.
- First, for each characteristic i, we design a *membership function* μ_i that assigns:
  - to each possible value x_i of this characteristic,
  - a degree – on the scale from 0 to 1 – to which this value satisfies the decision maker.

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6. Fuzzy decision making (cont-d)

- For each alternative $x = (x_1, \ldots, x_n)$:
  - we use the membership functions μ_i
  - to find the degrees $\mu_1(x_1)$, …, $\mu_n(x_n)$ to which the value of each of n characteristics is satisfactory.
- Then, we use an appropriate "and"-operation (t-norm) $f_\& : [0,1] \times [0,1] \to [0,1]$ to estimate:
  - the degree $\mu(x)$ to which the value of all n characteristics are satisfactory,
  - i.e., to which the entire alternative x is satisfactory:

$$\mu(x) = f_\&(\mu_1(x_1), \ldots, \mu_n(x_n)).$$

- After that, a reasonable idea is:
  - to select the alternative x
  - for which this overall degree of satisfaction $\mu(x)$ is the largest possible.

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7. This procedure is, of course, an approximation to the ideal exact decision making

- Of course, every time we use imprecise words, what we get is an approximate description.
- In the case of decision making, it is an approximate description of our preferences.
- There is a whole science of decision making that describes:
  - how to elicit exact preferences and
  - how to make exact decisions.
- A natural question is: how accurately does fuzzy decision making approximate the exact one?
- When can the actual decision making be approximated by a fuzzy process with any given accuracy?
- When it cannot be thus approximated?

8. Decision theory: a brief reminder

- The main objective of decision theory is to help people make reasonable decisions.
- Of course, people have different tastes and different preferences; so:
  - to be able to help a person (or a company) make a reasonable decision,
  - we first need to describe this person's preferences.

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9. How to elicit preferences: main idea

- A natural way to elicit information about a person's preferences is:
  - to provide this person with several real or hypothetical alternatives, and
  - to ask which of these alternatives this person prefers.
- For example, we can hypothetically propose to compare two houses on the same location:
  - with different prices and
  - different values of square footage.

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10. The idea of a complex alternative – i.e., an alternative with a probabilistic outcome

- We can also ask a person to compare more complex situations, in which this person
  - gets different alternatives
  - with some probabilities.
- This is not just a purely mathematical idea, this happens in real life all the time.
- Let us give a realistic example.
- Originally, a person had two alternatives:
  - a somewhat worse alternative x (e.g., a house too far away), and
  - a somewhat better alternative x'.
- The buyer is finishing negotiations with the owner of the house x'.

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11. The idea of a complex alternative (cont-d)

- The buyer is sure that the owner will prefer to sell:
  - to this buyer,
  - and not to this buyer's competitor (who is also willing to buy this house).
- However, the seller of the house x' is in a hurry, so the deal needs to be signed right away.
- Suddenly, the buyer learns that a new, even better, house x'' has just appeared on the market.
- For this house, there are already many bidders, so:
  - it is expected to take some time for the owner of this new house to decide whom to sell,
  - and, of course, there is no guarantee that our buyer will get this house.

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12. The idea of a complex alternative (cont-d)

- From the experience of a real estate agent, the buyer knows the probability p of winning the bid for x''.
- So, the buyer has two choices:
  - the buyer can ignore the new house and buy the house x';
  - alternatively, the buyer can abandon negotiations about the house x' and bid for the new house.
- In the second case:
  - with probability p, the buyer will win the bid and get the new house x'', and
  - with the remaining probability $1-p$, the buyer will get the only remaining alternative x.

13. The idea of a complex alternative (cont-d)

- Such situations are quite realistic, so it makes sense to ask a decision maker:
  - not only to compare original alternatives,
  - but also to compare such "complex" alternatives – i.e., alternatives with a probabilistic outcome.

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14. How to describe such complex alternatives in precise terms

- We have a finite list X of *actual* alternatives – e.g., actual houses that are or can be on the market.
- We can expand this list by adding finitely many *hypothetical* alternatives.
- *Example:*
  - ideal houses or
  - houses which are real but which we know will not be on the market, such as the White House.
- Let us denote this expanded list of alternatives by $\mathcal{X}$.
- A complex alternative means that we select some of the alternatives from the set $\mathcal{X}$ with different probabilities.
- In precise terms, a complex alternative is a probability measure on the set $\mathcal{X}$.

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15. Describing complex alternatives (cont-d)

- In decision theory, such complex alternatives are known as *lotteries*.
- The set of all such lotteries will be denoted by $\mathcal{L}$.
- Each alternative x from the extended list $\mathcal{X}$ can be identified with a complex alternative in which:
  - this alternative appears with probability 1, and
  - all other alternatives have probability 0.

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16. The idea of a complex alternative (cont-d)

- We assume that for every two lotteries L and L', the decision maker can decide:
  - whether the lottery L is better; we will denote this preference by $L' < L$;
  - or the lottery L' is better; we will denote this preference by $L < L'$;
  - or the lotteries L and L' are of the same value to the decision maker; we will denote this situation by

$$L \sim L'.$$

- We assume that the decision maker is rational.

17. The idea of a complex alternative (cont-d)

- This means, in particular, that:
  - if L' is better than L ($L < L'$) and L'' is better than L' ($L' < L''$),
  - then L'' should be better than L ($L < L''$).
- In mathematical terms, the preference relation must be *transitive.*
- Similarly, if $L < L'$ and $L' \sim L''$, then we must have $L < L''$, etc.

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18. A numerical scale for preferences

- Real-life alternative are rarely perfectly good, and rarely perfectly bad.
- As a result, in most practical situations, it is possible to add:
  - to the set X of actual alternatives,
  - the following two hypothetical alternatives.
- An alternative x_- which is worse than anything that we will actually encounter, i.e., worse than all $x \in X$.
- We will call this alternative *very bad.*
- An alternative x_+ which is better than anything that we will actually encounter, i.e., better than all $x \in X$.
- We will call this alternative *very good.*
- Thus, we get the set $\mathcal{X} = X \cup \{x_-, x_+\}$.

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19. A numerical scale for preferences (cont-d)

- For example, in the situation of buying a house:
  - the very good alternative x_+ may mean buying the White House, while
  - the very bad alternative x_- means staying is the same crammed apartment as before.
- (In principle, we can also add other hypothetical alternatives, i.e., get a set $\mathcal{X} \supset X \cup \{x_-, x_+\}$.)
- For each real number p from the interval $[0, 1]$, we can consider a lottery in which:
  - the probability of the very good alternative x_+ is equal to p,
  - the probability of the very bad alternative x_- is equal to $1 - p$, and
  - the probability of all other alternatives is 0.

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20. A numerical scale for preferences (cont-d)

- We will denote this lottery by $L(p)$.
- Now, we have a continuous family of lotteries characterized by the parameter p.
- Let us show how this family can be used to provide a numerical value to each alternative from the set X.
- More generally, we can provide a value to each lottery L in which we only alternatives from X.
- Let $x \in X$ be an alternative.
- Then, we can compare it with the lotteries $L(p)$ corresponding to different values p.
- For each p, we have either $L(p) < x$, or $L(p) \sim x$, or $x \sim L(p)$.
- *Example:* a house described by characteristics $x = (x_1, \ldots, x_n)$, $p = 0.4$.

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21. A numerical scale for preferences (cont-d)

- Then, we ask a person to compare the following two complex alternatives:
  - getting this house x, and
  - getting the White House with probability $p = 0.4$ and getting nothing with probability $1 - p = 0.6$.
- When $p = 0$, then the lottery $L(0)$ coincides with the very bad alternative x_-.
- Thus, because of our selection of the alternative x_-, we have $x_- < x$, i.e., $L(0) < x$.
- When $p = 1$, then the lottery $L(1)$ coincides with the very good alternative x_+.
- Thus, because of our selection of the alternative x_+, we have $x < x_+$, i.e., $x < L(1)$.

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22. A numerical scale for preferences (cont-d)

- Let us consider two such lotteries $L(p)$ and $L(q)$ with probabilities $p < q$.
- Then clearly a user will prefer the lottery $L(q)$ in which:
  - the probability of the very good alternative is higher, and
  - the probability of the very bad alternative is lower:

$$L(p) < L(q).$$

- Thus:
  - if $L(q) < x$ (or $L(q) \sim x$) and $p < q$, then we must have $L(p) < x$; and
  - if $x < L(p)$ (or $x \sim L(p)$) and $p < q$, then we must have $x < L(q)$.

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23. A numerical scale for preferences (cont-d)

- One can prove that $u(x) \stackrel{\text{def}}{=} \sup\{p : L(p) < x\}$ is equal to $\inf\{p : x < L(p)\}$, and that:
  - for all $p < u(x)$, we have $L(p) < x$, and
  - for all $p > u(x)$, we have $x < L(p)$.
- This "threshold" probability value $u(x)$ is known as the *utility* of the alternative x.
- By a *utility function*, we mean a mapping that assigns:
  - to some alternatives (actual, hypothetical, and/or complex),
  - the value $u(x)$.

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24. A numerical scale for preferences (cont-d)

- Strictly speaking, we have defined utility only for alternatives from the set X.
- However, a similar definition can be stated:
  - for complex alternatives L formed by actual alternatives (from X),
  - i.e., in mathematical terms, for probability distributions on the set X.

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25. Practical indistinguishability

- From the purely mathematical viewpoint:
  - lotteries $L(p)$ and $L(q)$ corresponding to different values $p \neq q$ are different,
  - even when these values are very close: e.g., when $p = 0.5$ and $q = 0.500001$.
- However, in practice, we will probably not notice this difference.
- Some folks *may* notice this difference.
- But they will not notice the difference between $p = 0.5$ and $q = 0.5 + \varepsilon$ for some positive value $\varepsilon \ll 0.000001$.
- There are two reasons for this.

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26. Practical indistinguishability (cont-d)

- Probabilities provided by the real estate agent:
  - based on a finite sample of cases and
  - are, thus, approximate.
- Even if we decide to use an actual fair coin to implement this lottery, this coin may be slightly flawed.
- Thus, its probability may be somewhat different from 0.5.
- In general:
  - for sufficiently small $\varepsilon > 0$,
  - we do not feel the difference between lotteries corresponding to probabilities p, $p - \varepsilon$, and $p + \varepsilon$.

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27. Practical indistinguishability (cont-d)

- By definition of the utility $u(x)$, for each $\varepsilon > 0$, we have $L(u(x) - \varepsilon) < x < L(u(x) + \varepsilon)$.
- Thus, from the practical viewpoint, the alternative x is actually equivalent to the lottery $L(u(x))$.
- We will denote this by $x \equiv L(u(x))$.

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28. Which alternative should we select?

- As we have mentioned:
  - each alternative x is equivalent to the lottery $L(u(x))$,
  - where $u(x)$ is the utility of this alternative.
- Thus, comparing alternatives is equivalent to comparing the corresponding lotteries $L(u(x))$.
- We have also mentioned that:
  - when we compare several lotteries $L(p)$, $L(q)$, $\ldots$,
  - then the larger the probability of the very good alternative x_+, the better.
- Thus, we have to select the alternative x for which the utility value $u(x)$ is the largest.

29. Why expected utility

- As we will show:
  - one of the consequences of the above definition of utility is that
  - in the case of uncertainty, we need to maximize the expected utility.
- Some folks – who are not very familiar with decision theory – mistakenly think that:
  - the maximization of expected utility
  - is an additional (and not-well-justified) postulate.
- But it is not, it is a consequence of utility's definition.

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30. Why expected utility (cont-d)

- Indeed, suppose that we have a complex alternative (lottery) L in which we get:
  - an alternative $x^{(1)} \in X$ with probability p_1, ...,
  - an alternative $x^{(m)} \in X$ with probability p_m,
  - and all other alternatives with probability 0.
- What is the utility $u(L)$ of this lottery?
- As we have mentioned, each alternative $x^{(i)}$ is equivalent to a lottery $L(u(x^{(i)}))$ in which we get:
  - the very good alternative x_+ with probability equal to the utility $u(x^{(i)})$ of this alternative, and
  - the very bad alternative x_- with the remaining probability $1 - u(x^{(i)})$.
- In the above lottery L, we can replace each alternative $x^{(i)}$ with the equivalent lottery $L(u(x^{(i)}))$.

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31. Why expected utility (cont-d)

- We conclude that the lottery L is equivalent to the following 2-stage lottery:
  - first, we select one of the alternatives $x^{(1)}, \ldots, x^{(m)}$, so that each $x^{(i)}$ is selected with prob. p_i.
  - then, depending on which alternative $x^{(i)}$ we selected, we select x_+ with prob. $u(x^{(i)})$ or x_-.
- As a result of this 2-stage lottery, we get either x_+ or x_-.
- There are m possible ways to get x_+, in each of which we get x_+ with the probability $p_i \cdot u(x^{(i)})$.
- So, the overall probability of selecting x_+ is equal to the sum of these values

$$s = p_1 \cdot u(x^{(1)}) + \ldots + p_m \cdot u(x^{(m)}).$$

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32. Why expected utility (cont-d)

- Thus, the lottery L is equivalent to the lottery $L(s)$ in which:
  - we get x_+ with probability s,
  - we get x_- with the probability $1 - s$,
  - and all other alternatives have probability 0.
- By definition of utility, this means that the lottery L has utility $u(L) = s$.
- It so happens that the expression for s is actually the expected value of the utility function $u(x)$.
- So, the principle of maximizing the expected utility indeed follows from the definition of utility.

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33. So when can exact decision be described in fuzzy terms?

- Now that we recalled the traditional decision theory, let us go back to the original question.
- In the general decision theory:
  - when we select between alternative x characterized by values $x_1, \ldots, x_n$,
  - we select the alternative $x^{\rm opt}$ for which the utility $u(x^{\rm opt}) = u(x_1^{\rm opt}, \ldots, x_n^{\rm opt})$ is the largest.
- In fuzzy decisions, we select the alternative for which the expression $\mu(x)$ attains its largest possible value:

$$\mu(x) = f_\&(\mu_1(x_1), \ldots, \mu_n(x_n)).$$

- Which preference relations can be represented as maximizing the expression $\mu(x)$ for some μ_i and $f_\&$?

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34. Let us reformulate fuzzy decision making in utility terms

- To be able to compare the two approaches, let us perform some reformulations.
- Specifically, we reformulate the fuzzy decision making in terms which are closer to utilities.
- In principle, there exist many different "and"-operations (t-norms).
- It is known that for every t-norm $f_\&$, for every $\varepsilon > 0$:
  - there exists a strictly increasing function $f : [0, 1] \to [0, 1]$ for which:
  - for all a and b, we have $|f_\&(a, b) - g(a, b)| \le \varepsilon$, where $g(a, b) \stackrel{\text{def}}{=} f^{-1}(f(a) \cdot f(b))$.

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35. Reformulating fuzzy decision making (cont-d)

- From the practical viewpoint, sufficiently close degrees of certainty are practically indistinguishable.
- Let us assume that we are asked to mark our degree of confidence on a scale from 0 to 5.
- We can definitely meaningfully distinguish between:
  - the value 0.6 (corresponding to selecting 3 on a 0 to 5 scale) and
  - the value 0.8 (corresponding to selecting 4 on this scale).
- Suppose now that a person asked to mark his/her degree of confidence on a scale from 0 to 100.
- Hardly anyone can distinguish between, e.g., marks 80 and 81 – which correspond to degree 0.80 and 0.81.

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36. Reformulating fuzzy decision making (cont-d)

- So, for small $\varepsilon > 0$ – e.g., for $\varepsilon \le 0.01$ – there is no difference between the degrees $f_\&(a, b)$ and $g(a, b)$.
- Thus, from the practical viewpoint, we can safely assume that the t-norm actually has the form

$$g(a, b) = f^{-1}(f(a) \cdot f(b)).$$

- For such a t-norm, the formula for $\mu(x)$ turns into

$$g(\mu_1(x_1), \ldots, \mu_n(x_n)) = f^{-1}(f(\mu_1(x_1)) \cdot \ldots \cdot f(\mu_n(x_n))).$$

- The function $f(x)$ is strictly increasing.
- So, maximizing this expression is equivalent to maximizing the result of applying f to this value:

$$f(\mu_1(x_1)) \cdot \ldots \cdot f(\mu_n(x_n)).$$

- Logarithm is also a strictly increasing function.

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37. Reformulating fuzzy decision making (cont-d)

- So, maximizing this product is equivalent to maximizing its logarithm.
- The logarithm of the product is equal to the sum of the logarithms:

$$v_1(x_1) + \ldots + v_n(x_n), \text{ where } v_i(x_i) \stackrel{\text{def}}{=} \ln(f(\mu_i(x_i))).$$

- Vice versa:
  - if our decision problem can be described in this form,
  - we can take, e.g., $f(x) = x$ (then $f_\&(a, b) = a \cdot b$), and $\mu_i(x_i) = C_i \cdot \exp(v_i(x_i))$, for some C_i;
  - C_i is needed to make sure that all the values of the resulting membership function do not exceed 1.

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38. Reformulating fuzzy decision making (cont-d)

- One can easily see that:
  - for these "and"-operation and membership functions,
  - maximizing $\mu(x)$ is indeed equivalent to maximizing the sum.

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39. Now the problem has been reformulated, so we can answer the original question

- The original problem was: when decision making can be described in fuzzy terms?
- Now we reformulate it in precise terms.
- When is:
  - a decision problem characterized by a utility function $u(x_1, \ldots, x_n)$
  - equivalent to maximizing the sum

$$v_1(x_1) + \ldots + v_n(x_n)?$$

- Interestingly, this problem has already been solved in utility theory.

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40. We can answer the original question (cont-d)

- Namely, one can easily check that:
  - if our decision making is equivalent to maximizing the sum,
  - this means that for us, the n characteristics are *independent* in the following sense.
- Suppose that we have two alternatives differing only by the values $x_i \neq x'_i$ of the i-th characteristic.
- Then which of these two alternative is better:
  - depends only on the values x_i and x'_i and
  - does not depend on the values of the other characteristics.

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41. We can answer the original question (cont-d)

- In precise terms:
  - if for some values $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$:

$$(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) < (x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n)$$

  - then for any other values $x_1', \ldots, x_{i-1}', x_{i+1}', \ldots, x_n'$, we will have a similar relation:

$$(x_1', \ldots, x_{i-1}', x_i, x_{i+1}', \ldots, x_n') < (x_1', \ldots, x_{i-1}', x_i', x_{i+1}', \ldots, x_n').$$

- This is true not only when we compare original alternatives.
- The same property holds if we consider complex alternatives (lotteries).

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42. We can answer the original question (cont-d)

- It has been proven that:
  - this independence property
  - uniquely characterizes the possibility of representation as the sum.
- So, we get the following answer to our questions.

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43. Conclusion

- We consider decision making problems, in which:
  - we compare alternatives x
  - characterized by the values of several characteristics $x_1, \ldots, x_n$.
- We show that:
  - such a decision problem can be represented in the equivalent fuzzy form
  - if and only if these characteristics are independent – in the above formal sense.

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44. Conclusion (cont-d)

- Independence means that when for two alternatives, all characteristics but one have equal values, then:
  - our preference depends only on the values of the differing characteristic and
  - does not depend on the values of all other characteristics.
- If this independence condition is satisfied, then fuzzy decision making:
  - can approximate the actual decision making as accurately as we want
  - it can even exactly represent the actual decision making.

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45. Conclusion (cont-d)

- On the other hand:
  - if the independence condition is not satisfied,
  - then the decision making cannot be exactly represented in the fuzzy form.
- Thus, there is a limit on how accurately fuzzy decision making can approximate this decision making.

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