How to Make Decision Under Interval Uncertainty: Description of All Reasonable Partial Orders on the Set of All Intervals

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1. Need to make decisions under uncertainty

- In many practical situations:
  - we have several alternatives to select from, and
  - we have an objective function that describes our preferences.
- For example, when we design an electric car:
  - we may want to maximize the distance that it can run until the next charge, or
  - we can minimize its weight, etc.
- In the ideal case, for each alternative, we know the exact value of the objective function.
- In this case:
  - If we want to maximize the objective function, we select the alternative with the largest value of this function.
  - If we want to minimize the objective function, we select the alternative with the smallest value of this function.

2. Need to make decisions under uncertainty (cont-d)

- However, in practice, we rarely know these exact values.
- What we usually know instead is the *interval* of possible values.
- It is therefore necessary to make a decision based on these interval values.
- In this talk, we analyze the problem of decision making under such interval uncertainty.
- This problem – as we will show – is already not easy.
- In practice, we may have experts describing us to what extent each of these values is possible; we hope that:
  - our analysis of decision making in the case of interval uncertainty
  - can help to make decisions under such fuzzy uncertainty.

3. Why decision making under interval uncertainty is not easy

- If we want to maximize the value of the objective function, then the value 4 is clearly better than the value 3.
- However, it is not clear whether, e.g., $[2, 5]$ corr. to Alternative 1 is better than $[3, 4]$ corr. to Alternative 2.
- Sometimes, a decision maker may have a clear preference between the two intervals.
- In other cases, the decision maker may be undecided.
- For real numbers, we have a linear (total) order.
- For every two different numbers either the first one is larger or the second one is larger,
- For comparing intervals, in general, we have *partial* order:
  - sometimes one interval is better than another interval, and
  - sometimes, there is no relation between them.

4. Decision making under interval uncertainty is not easy (cont-d)

- We want to help decision makers make good decisions in such situations.
- So, we need to know the partial order between intervals that describes the decision maker's preferences.
- In this talk, we describe all possible partial orders that satisfy some reasonable properties.

5. First reasonable property: additivity

- We will illustrate our analysis on simple financial examples, where we maximize the monetary gain.
- Suppose that the decision maker needs to decide between two alternatives characterized by the intervals $\mathbf{a} = [\underline{a}, \overline{a}]$ and $\mathbf{b} = [\underline{b}, \overline{b}]$.
- Gains rarely come from only one source, so:
  - in addition to the gain that will come from the decision maker selecting one of these two alternatives,
  - this decision maker may be bound to receive some additional amount resulting from his/her previous decisions.
- This additional amount may also not be known exactly, we may only know the interval $\mathbf{c} = [\underline{c}, \overline{c}]$ of possible gains.
- With this additional gain, we have two different ways to look at the original choice problem.

6. Additivity (cont-d)

- We can ignore this additional gain and consider it as the problem of selecting between the intervals $\mathbf{a}$ and $\mathbf{b}$.
- Alternatively, we can consider the overall gains of the decision maker in this situation.
- If the decision maker selects the alternative $\mathbf{a}$, then the possible values of his/her overall gain form an interval

$$\mathbf{a}+\mathbf{c}=\{a+c : a \in \mathbf{a} \text{ and } c \in \mathbf{c}\}.$$

- This interval is equal to $\mathbf{a}+\mathbf{c}=[\underline{a}+\underline{c}, \overline{a}+\overline{c}]$.
- Similarly, if the decision maker selects the alternative $\mathbf{b}$, then the possible values of his/her overall gain form an interval

$$\mathbf{b}+\mathbf{c}=[\underline{b}+\underline{c}, \overline{b}+\overline{c}].$$

- In this alternative description, we need to select between the two intervals $\mathbf{a}+\mathbf{c}$ and $\mathbf{b}+\mathbf{c}$.

7. Additivity (cont-d)

- These are two different descriptions of the exact same decision situation.
- So, it makes sense to require that the decision maker selects $\mathbf{b}$ over $\mathbf{a}$ if and only if he/she selects $\mathbf{b}+\mathbf{c}$ over $\mathbf{a}+\mathbf{c}$.
- A partial order $\preceq$ on the set of all intervals is called *additive* if for every three intervals $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, $\mathbf{a} \preceq \mathbf{b} \Leftrightarrow \mathbf{a}+\mathbf{c} \preceq \mathbf{b}+\mathbf{c}$.

8. Second reasonable property: antisymmetry

- Sometimes, we have zero-sum situations in which one party's gain is another party's loss.
- Let us consider the simplest case when the two parties has the same preferences; in this case:
  - if for the first party, the gain a is better than the gain b,
  - this would mean that for the second party, the loss of a should be worse than the loss of b.
- The loss of a is, in mathematical terms, the same as gain $-a$.
- Thus, the requirement is that if $a \le b$, then we should have $-b \le -a$.
- A partial order $\preceq$ on the set of all intervals is called *antisymmetric* if for every two intervals $\mathbf{a}$ and $\mathbf{b}$, we have $\mathbf{a} \preceq \mathbf{b} \Rightarrow -\mathbf{b} \preceq -\mathbf{a}$, where

$$-\mathbf{a} \stackrel{\text{def}}{=} \{-a : a \in \mathbf{a}\} = [-\overline{a}, -\underline{a}].$$

9. Third reasonable property: homogeneity

- In general, if the gain b is better than the gain a, then:
  - half of b is still better than half of a,
  - twice the gain of b is better than twice the gain of a, and,
  - in general, for every $\lambda > 0$, $\lambda \cdot b$ is better than $\lambda \cdot a$.
- It is reasonable to require the same property for intervals, where $\lambda \cdot \mathbf{a}$ means $\lambda \cdot [\underline{a}, \overline{a}] \stackrel{\text{def}}{=} \{\lambda \cdot a : a \in [\underline{a}, \overline{a}]\}$.
- A partial order $\preceq$ on the set of all intervals is called *homogeneous* if for every two intervals $\mathbf{a}$ and $\mathbf{b}$ and for every real number $\lambda > 0$:

$$\mathbf{a} \preceq \mathbf{b} \Rightarrow \lambda \cdot \mathbf{a} \preceq \lambda \cdot \mathbf{b}.$$

10. Main Result

- For every partial order $\preceq$ on the set of intervals, the following two conditions are equivalent to each other:
  - the order is additive, antisymmetric, and homogeneous;
  - there exist real numbers $\underline{c}_1, \overline{c}_1, \underline{c}_2, \overline{c}_2$ and relations $<_1, <_2 \in \{<, \preceq\}$ for which $[\underline{a}, \overline{a}] \preceq [\underline{b}, \overline{b}] \Leftrightarrow \underline{c}_1 \cdot \underline{a} + \overline{c}_1 \cdot \overline{a} <_1 \underline{c}_1 \cdot \underline{b} + \overline{c}_1 \cdot \overline{b}$ and
  $$\underline{c}_2 \cdot \underline{a} + \overline{c}_2 \cdot \overline{a} <_2 \underline{c}_2 \cdot \underline{b} + \overline{c}_2 \cdot \overline{b}.$$
- In each pair $(\underline{c}_i, \overline{c}_i)$, we can divide both sides of each inequality by the absolute value of a non-zero parameter.
- Thus, we get an equivalent inequality with only one parameter.
- An order is linear if and only if we have only one inequality.
- This case is known as Hurwicz optimism-pessimism criterion, a criterion that was awarded a Nobel prize.

11. Proof

- It is easy to check that every partial order than is described by the parameters $\underline{c}_i$ and $\overline{c}_i$ is additive, antisymmetric, and homogeneous.
- So, to complete the proof, it is sufficient to prove that every additive, antisymmetric, and homogeneous partial order $\preceq$ has this form.
- In the following proof:
  - we will assume that the order $\preceq$ has these three properties, and
  - we will show that it has the desired form.
- Let us first prove that the order $\preceq$ is uniquely determined by the set C of all the interval $\mathbf{a}$ for which $[0, 0] \preceq \mathbf{a}$.
- For this purpose, we will consider two possible cases:
  - the case when the width $\overline{b} - \underline{b}$ of the interval $\mathbf{b}$ is larger than or equal to the width $\overline{a} - \underline{a}$ of the interval $\mathbf{a}$, and
  - the case when the interval $\mathbf{a}$ has the larger width.

12. Proof (cont-d)

- In the first case, by adding $\underline{b} - \overline{a}$ to both sides of the inequality $\overline{b} - \underline{b} \geq \overline{a} - \underline{a}$, we conclude that $\overline{b} - \overline{a} \geq \underline{b} - \underline{a}$.
- Thus, we can have an interval $[\underline{b} - \underline{a}, \overline{b} - \overline{a}]$.
- We will denote this interval by $\mathbf{b} \ominus \mathbf{a}$.
- One can easily check that we have $\mathbf{b} = (\mathbf{b} \ominus \mathbf{a}) + \mathbf{a}$ and that we have $\mathbf{a} = [0, 0] + \mathbf{a}$.
- Thus, by additivity, we have $\mathbf{a} \preceq \mathbf{b} \Leftrightarrow [0, 0] \preceq \mathbf{b} \ominus \mathbf{a}$.
- So, by definition of the set C: $\mathbf{a} \preceq \mathbf{b} \Leftrightarrow \mathbf{b} \ominus \mathbf{a} \in C$.
- In the second case, by adding $\underline{a} - \overline{b}$ to both sides of the inequality $\overline{b} - \underline{b} < \overline{a} - \underline{a}$, we conclude that $\underline{a} - \overline{b} < \overline{a} - \overline{b}$.
- Thus, we can have an interval $\mathbf{a} \ominus \mathbf{b}$.
- One can easily check that we have $\mathbf{a} = (\mathbf{a} \ominus \mathbf{b}) + \mathbf{b}$ and that we have $\mathbf{b} = [0, 0] + \mathbf{b}$.

13. Proof (cont-d)

- Thus, by additivity, we have $\mathbf{a} \preceq \mathbf{b} \Leftrightarrow \mathbf{a} \ominus \mathbf{b} \preceq [0,0]$.
- Now, due to antisymmetry, the condition $\mathbf{a} \ominus \mathbf{b} \preceq [0,0]$ is equivalent to $[0,0] \preceq -(\mathbf{a} \ominus \mathbf{b}) = [\overline{b} - \overline{a}, \underline{b} - \underline{a}]$, i.e., to:

$$\mathbf{a} \preceq \mathbf{b} \Leftrightarrow [\overline{b} - \overline{a}, \underline{b} - \underline{a}] \in C.$$

- Thus, indeed, the order $\preceq$ is uniquely determined by the set C.
- Let us use a natural geometric representation of intervals to analyze the properties of the set C.
- Namely, each interval $[\underline{a}, \overline{a}]$ can be naturally represented by a planar point with coordinates $(\underline{a}, \overline{a})$.
- In this representation, the set C becomes a set of such points, i.e., a planar set.
- Due to homogeneity, if $[0,0] \preceq \mathbf{a}$, then for each $\lambda > 0$, we have $[0,0] \preceq \lambda \cdot \mathbf{a}$.

14. Proof (cont-d)

- In geometric terms, this means that the set C is closed under multiplication by a positive number.
- What if the set C contains two intervals $\mathbf{a}$ and $\mathbf{b}$, i.e., if $[0,0] \preceq \mathbf{a}$ and $[0,0] \preceq \mathbf{b}$.
- Then, by additivity, we get $[0,0]+\mathbf{b} \preceq \mathbf{a}+\mathbf{b}$, i.e., $\mathbf{b} \preceq \mathbf{a}+\mathbf{b}$.
- So, since partial order is transitive, we get $[0,0] \preceq \mathbf{a}+\mathbf{b}$.
- In geometric terms, this means that the set C is closed under addition.
- Thus, the set C is closed under multiplication by a positive constant and under addition.
- This means that the set C is a convex cone.
- It is known that all convex cones in a plane have the desired representation.
- Namely, they form a space between two lines – each of which is described by a homogeneous linear equation. Q.E.D.