How to Make Decision Under Interval Uncertainty: Description of All Reasonable Partial Orders on the Set of All Intervals

Tiago M. Costa\textsuperscript{a}, Olga Kosheleva\textsuperscript{b}, and Vladik Kreinovich\textsuperscript{c}

\textsuperscript{a}Dept. Matemática, Univ. de Tarapacá, Casilla 7D, Arica, Chile, grafunjo@yahoo.com.br
\textsuperscript{b,c}Departments of Teacher Education and Computer Science
University of Texas at El Paso, El Paso, Texas 79968, USA
olgak@utep.edu, vladik@utep.edu
1. Need to make decisions under uncertainty

- In many practical situations:
  - we have several alternatives to select from, and
  - we have an objective function that describes our preferences.
- For example, when we design an electric car:
  - we may want to maximize the distance that it can run until the next charge, or
  - we can minimize its weight, etc.
- In the ideal case, for each alternative, we know the exact value of the objective function.
- In this case:
  - If we want to maximize the objective function, we select the alternative with the largest value of this function.
  - If we want to minimize the objective function, we select the alternative with the smallest value of this function.
2. Need to make decisions under uncertainty (cont-d)

- However, in practice, we rarely know these exact values.
- What we usually know instead is the *interval* of possible values.
- It is therefore necessary to make a decision based on these interval values.
- In this talk, we analyze the problem of decision making under such interval uncertainty.
- This problem – as we will show – is already not easy.
- In practice, we may have experts describing us to what extent each of these values is possible; we hope that:
  - our analysis of decision making in the case of interval uncertainty
  - can help to make decisions under such fuzzy uncertainty.
3. Why decision making under interval uncertainty is not easy

- If we want to maximize the value of the objective function, then the value 4 is clearly better than the value 3.

- However, it is not clear whether, e.g., $[2, 5]$ corr. to Alternative 1 is better than $[3, 4]$ corr. to Alternative 2.

- Sometimes, a decision maker may have a clear preference between the two intervals.

- In other cases, the decision maker may be undecided.

- For real numbers, we have a linear (total) order.

- For every two different numbers either the first one is larger or the second one is larger,

- For comparing intervals, in general, we have *partial* order:
  - sometimes one interval is better than another interval, and
  - sometimes, there is no relation between them.
4. Decision making under interval uncertainty is not easy (cont-d)

- We want to help decision makers make good decisions in such situations.
- So, we need to know the partial order between intervals that describes the decision maker’s preferences.
- In this talk, we describe all possible partial orders that satisfy some reasonable properties.
5. First reasonable property: additivity

- We will illustrate our analysis on simple financial examples, where we maximize the monetary gain.
- Suppose that the decision maker needs to decide between two alternatives characterized by the intervals \( a = [a, \bar{a}] \) and \( b = [b, \bar{b}] \).
- Gains rarely come from only one source, so:
  - in addition to the gain that will come from the decision maker selecting one of these two alternatives,
  - this decision maker may be bound to receive some additional amount resulting from his/her previous decisions.
- This additional amount may also not be known exactly, we may only know the interval \( c = [\underline{c}, \bar{c}] \) of possible gains.
- With this additional gain, we have two different ways to look at the original choice problem.
6. **Additivity (cont-d)**

- We can ignore this additional gain and consider it as the problem of selecting between the intervals $a$ and $b$.
- Alternatively, we can consider the overall gains of the decision maker in this situation.
- If the decision maker selects the alternative $a$, then the possible values of his/her overall gain form an interval
  \[ a + c = \{a + c : a \in a \text{ and } c \in c\} \]
  This interval is equal to $a + c = [a + c, \bar{a} + \bar{c}]$.
- Similarly, if the decision maker selects the alternative $b$, then the possible values of his/her overall gain form an interval
  \[ b + c = [b + c, \bar{b} + \bar{c}] \]
- In this alternative description, we need to select between the two intervals $a + c$ and $b + c$. 

7. Additivity (cont-d)

- These are two different descriptions of the exact same decision situation.

- So, it makes sense to require that the decision maker selects $b$ over $a$ if and only if he/she selects $b + c$ over $a + c$.

- A partial order $\preceq$ on the set of all intervals is called additive if for every three intervals $a$, $b$, and $c$, $a \preceq b \iff a + c \preceq b + c$. 

8. Second reasonable property: antisymmetry

- Sometimes, we have zero-sum situations in which one party’s gain is another party’s loss.

- Let us consider the simplest case when the two parties have the same preferences; in this case:
  - if for the first party, the gain \( a \) is better than the gain \( b \),
  - this would mean that for the second party, the loss of \( a \) should be worse than the loss of \( b \).

- The loss of \( a \) is, in mathematical terms, the same as gain \(-a\).

- Thus, the requirement is that if \( a \leq b \), then we should have \(-b \leq -a\).

- A partial order \( \preceq \) on the set of all intervals is called \textit{antisymmetric} if for every two intervals \( a \) and \( b \), we have \( a \preceq b \Rightarrow -b \preceq -a \), where

\[
-a \overset{\text{def}}{=} \{-a : a \in a\} = [-\bar{a}, -\underline{a}].
\]
9. Third reasonable property: homogeneity

- In general, if the gain $b$ is better than the gain $a$, then:
  - half of $b$ is still better than half of $a$,
  - twice the gain of $b$ is better than twice the gain of $a$, and,
  - in general, for every $\lambda > 0$, $\lambda \cdot b$ is better than $\lambda \cdot a$.

- It is reasonable to require the same property for intervals, where $\lambda \cdot a$ means $\lambda \cdot [a, \bar{a}] \overset{\text{def}}{=} \{ \lambda \cdot a : a \in [a, \bar{a}] \}$.

- A partial order $\leq$ on the set of all intervals is called *homogeneous* if for every two intervals $a$ and $b$ and for every real number $\lambda > 0$:

$$a \preceq b \Rightarrow \lambda \cdot a \preceq \lambda \cdot b.$$
10. Main Result

- For every partial order \( \preceq \) on the set of intervals, the following two conditions are equivalent to each other:
  - the order is additive, antisymmetric, and homogeneous;
  - there exist real numbers \( c_1, \bar{c}_1, c_2, \bar{c}_2 \) and relations \( <_1, <_2 \in \{<, \preceq\} \) for which \([a, \bar{a}] \preceq [b, \bar{b}] \iff c_1 \cdot a + \bar{c}_1 \cdot \bar{a} <_1 c_1 \cdot b + \bar{c}_1 \cdot \bar{b} \) and \( c_2 \cdot a + \bar{c}_2 \cdot \bar{a} <_2 c_2 \cdot b + \bar{c}_2 \cdot \bar{b} \).

- In each pair \((c_i, \bar{c}_i)\), we can divide both sides of each inequality by the absolute value of a non-zero parameter.

- Thus, we get an equivalent inequality with only one parameter.

- An order is linear if and only if we have only one inequality.

- This case is known as Hurwicz optimism-pessimism criterion, a criterion that was awarded a Nobel prize.
11. Proof

- It is easy to check that every partial order than is described by the parameters $c_i$ and $\overline{c}_i$ is additive, antisymmetric, and homogeneous.

- So, to complete the proof, it is sufficient to prove that every additive, antisymmetric, and homogeneous partial order $\preceq$ has this form.

- In the following proof:
  - we will assume that the order $\preceq$ has these three properties, and
  - we will show that it has the desired form.

- Let us first prove that the order $\preceq$ is uniquely determined by the set $C$ of all the interval $a$ for which $[0, 0] \preceq a$.

- For this purpose, we will consider two possible cases:
  - the case when the width $\overline{b} - b$ of the interval $b$ is larger than or equal to the width $\overline{a} - a$ of the interval $a$, and
  - the case when the interval $a$ has the larger width.
12. Proof (cont-d)

- In the first case, by adding $b - a$ to both sides of the inequality $\bar{b} - \bar{b} \geq \bar{a} - a$, we conclude that $\bar{b} - \bar{a} \geq b - a$.

- Thus, we can have an interval $[b - a, \bar{b} - \bar{a}]$.

- We will denote this interval by $b \ominus a$.

- One can easily check that we have $b = (b \ominus a) + a$ and that we have $a = [0, 0] + a$.

- Thus, by additivity, we have $a \preceq b \iff [0, 0] \preceq b \ominus a$.

- So, by definition of the set $C$: $a \preceq b \iff b \ominus a \in C$.

- In the second case, by adding $a - \bar{b}$ to both sides of the inequality $\bar{b} - \bar{b} < \bar{a} - a$, we conclude that $a - \bar{b} < \bar{a} - \bar{b}$.

- Thus, we can have an interval $a \ominus b$.

- One can easily check that we have $a = (a \ominus b) + b$ and that we have $b = [0, 0] + b$. 
13. Proof (cont-d)

- Thus, by additivity, we have \( a \preceq b \iff a \ominus b \preceq [0,0] \).

- Now, due to antisymmetry, the condition \( a \ominus b \preceq [0,0] \) is equivalent to \([0,0] \preceq -(a \ominus b) = [\bar{b} - \bar{a}, b - a], \) i.e., to:
  \[
  a \preceq b \iff [\bar{b} - \bar{a}, b - a] \in C.
  \]

- Thus, indeed, the order \( \preceq \) is uniquely determined by the set \( C \).

- Let us use a natural geometric representation of intervals to analyze the properties of the set \( C \).

  - Namely, each interval \([a, \bar{a}]\) can be naturally represented by a planar point with coordinates \((a, \bar{a})\).

  - In this representation, the set \( C \) becomes a set of such points, i.e., a planar set.

  - Due to homogeneity, if \([0,0] \preceq a\), then for each \( \lambda > 0 \), we have \([0,0] \preceq \lambda \cdot a\).
14. Proof (cont-d)

- In geometric terms, this means that the set $C$ is closed under multiplication by a positive number.

- What if the set $C$ contains two intervals $a$ and $b$, i.e., if $[0,0] \preceq a$ and $[0,0] \preceq b$.

- Then, by additivity, we get $[0,0] + b \preceq a + b$, i.e., $b \preceq a + b$.

- So, since partial order is transitive, we get $[0,0] \preceq a + b$.

- In geometric terms, this means that the set $C$ is closed under addition.

- Thus, the set $C$ is closed under multiplication by a positive constant and under addition.

- This means that the set $C$ is a convex cone.

- It is known that all convex cones in a plane have the desired representation.

- Namely, they form a space between two lines – each of which is described by a homogeneous linear equation. Q.E.D.