Fuzzy Mathematics under Non-Minimal "And"-Operations (t-Norms): Equivalence leads to Metric, Order Leads to Kinematic Metric, Topology Leads to Area or Volume

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1. Fuzzy mathematics: a brief reminder

- Usually, fuzzy techniques are used to describe our uncertainty about the numerical values.
- However, people also use fuzzy words to describe more complex mathematical objects.
- For example, an engineer can say that:
  - the function $y = f(x)$ describing the dependence between the two quantities is "rather smooth" or "very smooth",
  - or that the solution to the corresponding system of equation is "usually unique", etc.
- To formalize such use of "fuzzy" natural-language words, researchers have extended fuzzy techniques to more general *fuzzy mathematics*.
- There, fuzzy degrees can be applied to general mathematical objects.

2. Fuzzy mathematics mostly uses the min t-norm

- Mathematics is not easy, and extending mathematics to fuzzy objects does not make it easier.
- Because of this complexity, most results of fuzzy mathematics limit themselves to the simplest t-norm, i.e., the min t-norm.
- A natural question is: what will happen if we use more general "and"-operations?
- In this talk, we show, on several examples, that the use of non-minimal "and"-operations naturally leads to new mathematical concepts:
  - equivalence leads to metric,
  - order leads to kinematic metric, and
  - topology leads to volume.

3. What is equivalence: reminder

- In mathematics, a relation $\sim$ is called an *equivalent relation* if it satisfies the following three conditions:
  - it is *reflexive*, i.e., $x \sim x$ for all x;
  - it is *symmetric*, i.e., $x \sim y$ is equivalent to $y \sim x$ for all x and y, and
  - it is *transitive*, i.e., if $x \sim y$ and $y \sim z$, then $x \sim z$.
- In 2-valued logic, for every two objects x and y, either x is equivalent to y or not.
- A natural fuzzy analogue is when for every two objects x and y, we have a *degree* $d(x, y) \in [0, 1]$ to which these two objects are equivalent.
- Let us reformulate the above three properties in these terms.
- Reflexivity means that each object is absolutely equivalent to itself, i.e., that $d(x, x) = 1$ for all x.

4. Equivalence (cont-d)

- Symmetry means that $d(x, y) = d(y, x)$ for all x and y.
- Finally, transitivity means that our degree of belief that x is equivalent to z should be at least a large as our belief in the statement

  "x is equivalent to y and y is equivalent to z".

- So, we should have $d(x, z) \geq f_{\&}(d(x, y), d(y, z))$ for all x, y, and z.
- It is known that for every $\varepsilon > 0$, every "and"-operation can be approximated, with accuracy ε,
  - by a *strictly Archimedean "and"-operation*,
  - i.e., by an "and"-operation of the type $f_{\&}(a, b) = \psi^{-1}(\psi(a) \cdot \psi(b))$ for some strictly increasing 1-1 function $\psi : [0, 1] \mapsto [0, 1]$.

5. Fuzzy version naturally leads to metric

- In practice, we always get the degrees with some accuracy.
- Thus, from the practical viewpoint, it is safe to assume that the actual "and"-operation is strictly Archimedean.
- For such an "and"-operation, transitivity means that

$$d(x,z) \geq \psi^{-1}(\psi(d(x,y)) \cdot \psi(d(y,z))).$$

- Applying the increasing 1-1 function $\psi(a)$ to both sides of this inequality, we will get an equivalent inequality

$$\psi(d(x,z)) \geq \psi(d(x,y)) \cdot \psi(d(y,z)).$$

- Applying $-\ln(x)$ to both sides, for $\rho(x,y) \stackrel{\text{def}}{=} -\ln(\psi(d(x,y)))$, we get $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$, i.e., the usual metric.

6. Strict order relation: a brief reminder

- Let us consider a *strict order* relation, i.e., a relation $<$ which is:
  - *anti-reflexive*: $x \not< x$ for all x;
  - *anti-symmetric*: if $x < y$, then $y \not< x$; and
  - *transitive*: if $x < y$ and $y < z$, then $x < z$.
- An example of such a relation is "x is smaller than y".
- We can have cases when x slightly smaller than y.
- We can have cases when x much smaller than y: $x \ll y$.
- The relation "much smaller" has the following natural property: if $x < y < z$, then:
  - whenever x is much smaller than y or y is much smaller than z,
  - then x is much smaller than z.

7. Strict order (cont-d)

- I.e., if $x < y < z$ and either $x \ll y$ or $y \ll z$, then $x \ll z$.
- To describe the relation $\ll$ in precise terms:
  - to each pair of objects x and y for which $x < y$,
  - we assign a degree $d(x, y)$ to which x is much smaller than y.
- When x is not smaller than y, then, of course, $d(x, y) = 0$.
- In terms of this degree, the above "or"-property, in fuzzy terms, takes the following form:

$$\text{if } x < y < z \text{ then } d(x, z) \geq f_\vee(d(x, y), d(y, z)).$$

- Each "or"-operation has the form $f_\vee(a, b) = 1 - f_\&(1 - a, 1 - b)$ for some "and"-operation $f_\&(a, b)$, so:

8. Strict order leads to kinematic metric

- Substituting this expression into the above inequality, we conclude that $d(x,z) \geq 1 - f_{\&}(1 - d(x,y), 1 - d(y,z))$.
- Subtracting both sides of this inequality from 1, we conclude that

$$1 - d(x,z) \leq f_{\&}(1 - d(x,y), 1 - d(y,z)).$$

- Similarly to the previous section, we can simplify this inequality if we apply $-\ln(\psi(a))$ to both sides.
- Then, for the values $\tau(x,y) \stackrel{\text{def}}{=} -\ln(\psi(1 - d(x,y))$, we get

$$\tau(x,z) \geq \tau(x,y) + \tau(y,z).$$

- This inequality is known as the *anti-triangle inequality.*
- It is true in physics, where $\tau(x,y)$ indicates the longest time needed to go from space-time event x to space-time event y.
- This longest time corresponds to rest or inertial motion.

9. Strict order leads to kinematic metric (cont-d)

- From this viewpoint, this inequality described the so-called *twin paradox* of relativity theory:
  - a twin that stays on Earth – and thus, stays on the rest trajectory from x to z – grows older than
  - the twin that first traveled to a faraway star (y) and then came back.
- In general, functions $\tau(x, y)$ that satisfy this inequality as known as *kinematic metrics.*
- They form the basis of the study of space-time.
- So, fuzzification of order indeed leads to a kinematic metric.

10. Third Example: Topology Leads to Area or Volume

- In mathematics, topology is usually defined as:
  - a class of open sets, i.e.,
  - class of sets that contain, with each point, some neighborhood of this point.
- One of the most frequent ways to define topology is to describe its *basis*, i.e., describe a class of open sets – e.g., on a plane, small circles.
- Then, we say that a set is open if it is a union of some sets from the basis - or of their finite intersections.
- To analyze the possible use of fuzzy, let us consider the simplified case when:
  - we have a grid formed by points with integer coordinates $c = (c_1, \ldots, c_n)$, and
  - the basis consists of semi-open boxes attached to each point c.

11. Topology (cont-d)

- For each grid point c, the corresponding box $B(c)$ consists of all the points $x = (x_1, \ldots, x_n)$ for which $c_i \le x_i < c_i + 1$ for all i.
- In this scheme, the only open sets are finite unions of such boxes.

12. What fuzzy adds to this description

- One of the main ideas behind fuzzy techniques is that everything is a matter of degree.
- In particular, this means that the boxes are not *absolutely* open, but rather open *with some degree of confidence* d.
- In this case, the union $S = B_1 \cup \ldots \cup B_k$ of k boxes B_i is open if and only if
  - each of these k boxes is open, i.e.,
  - if the box B_1 is open, *and* the box B_2 is open, etc.
- For each of these k statements, the degree of confidence is d.
- Thus, our degree of confidence $d(S)$ that all k boxes are open can be obtained by applying the "and"-operation to these k values:

$$d(S) = f_{\&}(d, \ldots, d) \text{ (k times).}$$

13. What fuzzy adds to this description (cont-d)

- If we now substitute the expression for the generic "and"-operation, we thus conclude that $d(S) = \psi^{-1}(\psi(d) \cdot \ldots \cdot \psi(d))$ (k times), i.e.,

$$d(S) = \psi^{-1}((\psi(d))^k).$$

- If we apply ψ to both sides, we get $\psi(d(S)) = (\psi(d))^k$.
- If we now apply the function $-\ln(a)$ to both sides, then for the resulting expression $V(S) \stackrel{\text{def}}{=} -\ln(\psi(d(S)))$, we get:

$$V(S) = k \cdot V_0, \text{ where } V_0 \stackrel{\text{def}}{=} -\ln(\psi(d)).$$

- So, by fuzzifying topology, we indeed get a value which is proportional:
  - to the number of boxes forming the set S,
  - i.e., depending on the dimension, to the area or to the volume of this set.
- If we decrease the size of the boxes, then, in the limit when this size tends to 0, we get the actual area or volume.

14. What fuzzy adds to this description (cont-d)

- In our description, we simplified this example so that computations will be clear.
- In doing this, we uses boxes which are *not* actually open in the usual Euclidean topology.
- However, a similar result holds if we use intersecting open small boxes instead.