Which Fuzzy Implication Operations Are Polynomial? A Theorem Proves That This Can Be Determined by a Finite Set of Inequalities

Sebastia Massanet[a], Olga Kosheleva[b], and Vladik Kreinovich[c]

[a]Department of Mathematics and Computer Science, University of Balearic Islands
07122 Palma, Balearic Islands, Spain, s.massanet@uib.es
[b,c]Departments of [b]Teacher Education and [c]Computer Science
University of Texas at El Paso, El Paso, Texas 79968, USA
olgak@utep.edu, vladik@utep.edu

1. Need to approximate fuzzy operations

- Fuzzy operations should reflect the expert reasoning.
- The more accurately these operation reflect expert reasoning, the better we capture the expert knowledge.
- We should therefore be prepared to capture the actual human reasoning as accurately as possible – whatever the reasoning will be.
- Technique for capturing all kinds of empirical dependence is well known in science.
- For example, in physics the usual way to capture such a dependence is based on the fact that many dependencies are analytical.
- So they can be expanded into an infinite Taylor series.
- For functions of one variable, we have

$$f(x) = a_0 + a_1 \cdot x + a_2 \cdot x^2 + \ldots$$

2. Need to approximate fuzzy operations (cont-d)

- For functions of two variables, we have $f(x, y) = a_{00} + a_{10} \cdot x +$
$$a_{01} \cdot y + a_{20} \cdot x^2 + a_{11} \cdot x \cdot y + a_{02} \cdot y^2 + \ldots$$
- To get a good approximation, we keep only the few first terms in this expansion.
- In the first approximation, we only keep linear terms, i.e., approximate the desired function by a linear polynomial
$$f_1(x, y) = a_{00} + a_{01} \cdot x + a_{10} \cdot y.$$
- To get a more accurate approximation, we also keep quadratic terms, i.e., approximate the desired function by a quadratic polynomial
$$f_2(x, y) = a_{00} + a_{10} \cdot x + a_{01} \cdot y +$$
$$a_{20} \cdot x^2 + a_{11} \cdot x \cdot y + a_{02} \cdot y^2.$$
- To get an even more accurate approximation, we can add cubic terms, i.e., approximate the desired function by a cubic polynomial, etc.

3. Which of these polynomial functions are fuzzy operations: a challenge

- We want an operation that satisfies the corresponding properties.
- E.g., we want an "and"-operation to be commutative and associative.
- So, when we try to match the empirical data about these operations – coming from experts:
  - we should not consider all possible polynomial functions,
  - we should only consider functions that satisfy the corresponding properties.
- So:
  - among all the tuples coefficients a_i of the corresponding polynomials for which the resulting polynomial is a fuzzy operation,
  - we need to select the tuple that best fits the available data.
- The problem is that the condition "is a fuzzy operation" is very complicated.

4. Which of these polynomial functions are fuzzy operations: a challenge (cont-d)

- It means that some equality should be satisfied for all possible values x and y (or even x, y, and z).
- Since there are infinitely many possible values x and y, we thus have infinitely many constraints that need to be satisfied.
- There exist numerical (and even analytical) methods for optimizing a function under a finite number of constraints.
- However, there are, in general, no available methods for optimizing a function under infinitely many constraints.
- An ideal solution to this problem would be to translate:
  - the complex constraint – describing that a polynomial function is the corresponding fuzzy operation,
  - into a finite sequence of inequalities.

5. What is known: a brief (incomplete) overview

- Such a translation into a finite set of inequalities is known for several operations and for several degrees of the corresponding polynomial.
- For example, for polynomial degrees up to order 4, such a translation is known for describing:
  - whether a polynomial function is a fuzzy implication, and
  - whether it satisfies some additional properties.
- Implication operation is defined as a function $f(x,y)$ for which $f(0,0)=f(1,1)=1$, $f(1,0)=0$, and for which the following holds:

$$\forall x\,\forall y\,\forall z\,(x\le y\Rightarrow f(x,z)\ge f(y,z))\text{ and}$$

$$\forall x\,\forall y\,\forall z\,(y\le z\Rightarrow f(x,y)\ge f(x,z)).$$

6. What is known and what we do

- Finite sets of inequalities are also described for each of the following additional properties: $\forall y\,(f(1,y)=y)$; $\forall x\,(f(x,x)=1)$;

$$\forall x\,\forall y\,\forall z\,(f(x,f(y,z))=f(y,f(x,z)));\quad \forall x\,\forall y\,(x\le y\Leftrightarrow f(x,y)=1);$$

$$\forall x\,\forall y\,(f(x,f(x,y))=f(x,y)).$$

- What is *not* clear is which properties of fuzzy operations can be translated into an equivalent finite set of inequalities.
- We prove that a translation into an equivalent finite set of inequalities is always possible.
- We also provide an algorithm for this translation.
- Caution: the algorithm is not always practically useful: in some cases, it requires double exponential time.

7. Definition

- Let $c_1, \ldots, c_m$ be symbols for real-valued constants; let $n_1, \ldots, n_k$ be positive integers.
- Let $f_1, \ldots, f_k$ be symbols of functions, so that each f_i is a function of n_i real variables. Let $x, x_1, \ldots, y, z, \ldots$ be real-valued variables.
- A *term* is defined as follows:
  - every symbol c_i and evert variable x_i is a term;
  - if $t_1, \ldots, t_{n_i}$ are terms, then $f_i(t_1, \ldots, t_{n_i})$ is a term.
- A *formula* is defined as follows:
  - if t and t' are terms, then $t = t'$, $t < t'$, $t > t'$, $t \leq t'$, $t \geq t'$, and $t \neq t'$ are formulas;
  - if F and G are formulas, then $F \,\&\, G$, $F \vee G$, $F \Rightarrow G$, $F \Leftrightarrow G$, and $\neg F$ are formulas;
  - if F is a formula, then $\forall x\, F$ and $\exists x\, F$ are formulas.
- A formula is *closed* is every variable is covered by some quantifier.

8. Definition (cont-d)

- Examples: $f(x, f(y, z))$ and $f(t(x, y), z)$ are terms, and all the above-described properties are closed formulas.
- By a polynomial equality, we mean an expression of the type $P = Q$, where P and Q are polynomials.
- A *polynomial inequality* is an expression of the type $P < Q$, $P > Q$, $P \leq Q$, $P \geq Q$, or $P \neq Q$, where P and Q are polynomials.
- By a *system of polynomial equalities and inequalities*, we mean a finite set of polynomial equalities and inequalities.
- We say that a tuple *satisfies* this system if it satisfies all equalities and inequalities from this system.
- By a *set of systems of polynomial equalities and inequalities*, we mean a finite set of systems of polynomial equalities and inequalities.
- We say that a tuple *satisfies* this set if it satisfies one of the systems from this set.

9. Definition (cont-d)

- *Example:* One of the conditions from the 2022 paper has the form

  If $\alpha < 0$ and $-2\alpha \le \beta - \delta \le -4\alpha$, then $(\beta-\delta)^2 + 2\alpha \cdot (\delta + \varepsilon + 1) \le 0$.

- In general, implication "if A then B" means that either B is true or A is false. So:
  - either we have $(\beta-\delta)^2 + 2\alpha \cdot (\delta + \varepsilon + 1) \le 0$
  - or the following condition is false:

  $$\text{“}\alpha < 0 \text{ and } -2\alpha \le \beta - \delta \text{ and } \beta - \delta \le -4\alpha\text{”}$$

- In general, the fact that the conjunction "A and B and C" is false means that either A is false, or B is false, or C is false.

- In our case, either $\alpha \ge 0$ or $-2\alpha > \beta - \delta$ or $\beta - \delta > -4\alpha$.

10. Definition (cont-d)

- Thus, the above condition means that the tuple $(\alpha, \beta, \ldots)$ satisfies the following set of four single-inequality systems:
  - $(\beta - \delta)^2 + 2\alpha \cdot (\delta + \varepsilon + 1) \le 0$;
  - $\alpha \ge 0$;
  - $-2\alpha > \beta - \delta$;
  - $\beta - \delta > -4\alpha$.
- Similarly, all other known conditions can be described in these terms.
- Let $x_1, x_2, \ldots$ be variables.
- A *monomial* is an expression of the type $x_{i_1}^{v_1} \cdot \ldots \cdot x_{i_m}^{v_m}$, where $i_1 < \ldots < i_m$ and v_i are positive integers.
- The sum $v_1 + \ldots + v_m$ is called the *degree* of the monomial.

11. Definitions and main result

- By a *polynomial of degree* d, we means an expression of the type $a_1 \cdot M_1 + \ldots + a_N \cdot M_N$, where M_i are of degree d.
- Let $d_1, \ldots, d_k$ be positive integers. Then:
- For each closed formula F, this formula, when limited to the case when each function f_i is a polynomial of degree d_i:
  - is equivalent to a set of systems of polynomial inequalities
  - in terms of constants c_i and coefficients of the polynomials f_i.
- There exists an algorithm that:
  - given a closed formula F – limited to the case when each function f_i is a polynomial of degree d_i,
  - returns the equivalent set of systems of polynomial equalities and inequalities.
- Same holds for rational and algebraic functions f_i of degree d_i.

12. Proof

- Our proof is based on the Tarski-Seidenberg theorem about the so-called first order theory of real numbers.
- We will denote this theory by T, after Tarski.
- In this theory T, we start with real-valued constants x_i and real-valued variables x_j.
- T-terms are polynomials in terms of c_i and x_j.
- Elementary T-formulas are equalities and inequalities between terms.
- A general T-formula is a formula obtained from elementary T-formulas by using:
  - logical connectives &, $\vee$, etc., and
  - quantifiers $\forall x$ and $\exists x$ over real numbers.

13. Proof (cont-d)

- The Tarski-Seidenberg theorem states that:
  - each closed T-formula F is equivalent to a set of systems of polynomial equalities and inequalities in terms of constants c_i; and
  - there exists an algorithm that, given a closed T-formula F, returns the equivalent set of systems of polynomial inequalities.
- To apply this theorem to our case, we need to represent our formulas in terms of the theory T.
- By comparing our definition of a formula with the definition of the T-formula, one can see that the only difference is:
  - between terms as defined above
  - and T-terms.
- So, the only thing we need for such a translation is to transform each term into the T-term form, i.e., into a polynomial.

14. Proof (cont-d)

- This can be done by induction over the inductive definition of a term:
- Constants and variables are already polynomials.
- If the terms $t_1, \ldots, t_{n_i}$ are polynomials, and the function f_i is a polynomial, then the expression $f_i(t_1, \ldots, t_{n_i})$ is also a polynomial.
- For example, if $f(x, y) = a_0 + a_1 \cdot x + a_1 \cdot y$, then

$$f(x, f(x, y)) = a_0 + a_1 \cdot x + a_2 \cdot (a_0 + a_1 \cdot x + a_2 \cdot y)) =$$

$$a_0 + a_1 \cdot x + a_2 \cdot x + a_2 \cdot a_0 + a_2 \cdot a_1 \cdot x + a_2 \cdot a_2 \cdot y.$$

- The proposition is thus proven.