Natural Color Interpretation of Interval-Valued Fuzzy Degrees

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1. Fuzzy knowledge is important

- A significant part of human knowledge is described by imprecise ("fuzzy") words from natural language.
- This includes expert knowledge in many application areas.
- For example, medical instructions are filled with words like "high fever", "high blood pressure", "low red blood cell count".
- There are often precise thresholds – e.g., the precise 38° C threshold for fever.
- However, it is very clear that there is no critical difference between temperatures 37.9 and 38.0.
- As a result, in such borderline cases, medical doctors do not blindly follow the formalized rules.
- Instead, they take into account the whole state of the patient when prescribing treatment.

2. Traditional $[0,1]$-based approach to describing fuzzy knowledge

- We need to describe imprecise (fuzzy) knowledge in computer-understandable numerical terms.
- Lotfi Zadeh suggested to describe:
  - for each corresponding statement,
  - the degree to which this statement is true.
- For example:
  - 38.5 is clearly a fever (with degree 1),
  - 36.6 is clearly not a fever (i.e., a fever with degree 0), while
  - 37.8 is a fever with some degree intermediate between 0 and 1.
- This idea started the currently well developed area of fuzzy techniques.
- This area has many successful applications.

3. Need to go beyond the traditional $[0,1]$-based logic

- In the traditional fuzzy technique:
  - to each imprecise natural-language property P (like "high") and
  - to each possible value x of the corresponding quantity (e.g., "temperature"),
  - we ask the expert to assign a degree $P(x)$ to which the value x has the property P.
- This idea works successfully in many applications.
- However, it does not fully capture our intuitive idea of fuzziness and imprecision:
- We start with the correct idea that users *cannot* come up with an *exact* threshold separating:
  - what is high and
  - what is not high.

4. Need to go beyond the $[0,1]$-based logic (cont-d)

- But then we require the same users to come up with the *exact* number describing their degree of certainty.
- Intuitively:
  - we can usually meaningfully distinguish between degrees 0.6 and 0.8,
  - but hardly anyone can meaningfully distinguish between degrees 0.8 and 0.81.
- It seems more natural to take into account that:
  - we can only assign degrees with uncertainty;
  - e.g., describe the range (interval) of possible degrees.
- If we allow intervals of degrees helps, this helps resolve another problems of $[0,1]$-based degree.

5. Need to go beyond the $[0,1]$-based logic (cont-d)

- For these degrees, there is no easy way to distinguish between:
  - the case when we have no information about the situation and
  - the case when we have many arguments in favor of the statements and equally many arguments against.
- In the $[0,1]$-based approach, both cases are described by the same degree 0.5.
- This is the degree which is exactly in the middle between 0 and 1.
- In contrast, in the interval-valued case we can do better.
- We can keep the degree 0.5 (i.e., the interval $[0.5, 0.5]$) for the case when we have:
  - many arguments for and
  - equally many arguments against.

6. Need to go beyond the $[0, 1]$-based logic (cont-d)

- And we can assign the interval $[0, 1]$ to the case when we have no information.
- This would mean that anything is possible.

7. If interval-valued fuzzy techniques are so promising, why not everyone is using them?

- On a theoretical level, interval-valued techniques provide a more adequate description of uncertainty.
- However, in practice, most applications of fuzzy techniques still use the traditional $[0, 1]$-based values.
- Why?
- A possible answer comes from the following.
- People are accustomed to marking a value on a scale:
  - we do it for surveys,
  - students do it for faculty evaluations,
  - we do it to indicate our opinion about the quality of a restaurant or a movie.

8. If interval-valued fuzzy techniques are so promising, why not everyone is using them (cont-d)

- However, most people are not accustomed to marking intervals.
- So, we need to make interval-valued techniques more natural, more convenient for people to use.

9. Geometric analysis of the problem

- In the traditional approach to fuzzy uncertainty:
  - possible degrees of certainty are
  - numbers from the interval $[0,1]$.
- In geometric terms, this interval can be described as follows:
  - we start with two degrees 0 and 1, and
  - we consider all possible convex combinations of these two degrees, i.e., all possible combinations of the type

$$\alpha_1 \cdot 1 + \alpha_0 \cdot 0, \text{ where } \alpha_i \geq 0 \text{ and } \alpha_0 + \alpha_1 = 1.$$

- Indeed, any value a from the interval $[0,1]$ can be represented in this form, with $\alpha_0 = a$ and $\alpha_1 = 1 - a$.

10. Geometric analysis of the problem (cont-d)

- Interval-valued degrees can be described in a similar way:
  - we have three degrees: 0 (i.e., $[0, 0]$), 1 (i.e., $[1, 1]$), and $[0, 1]$, and
  - we consider all possible convex combinations of these three degrees,
  - i.e., all possible combinations of the type

$$\alpha_1 \cdot [1, 1] + \alpha_0 \cdot [0, 0] + \alpha_{01} \cdot [0, 1]$$

$$\text{where } \alpha_i \geq 0 \text{ and } \alpha_0 + \alpha_1 + \alpha_{01} = 1.$$

- In this formula, addition and multiplication are understood component-wise, i.e.,
  - $\alpha \cdot [a, b]$ is understood as $[\alpha \cdot a, \alpha \cdot b]$, and
  - $[a_1, b_1] + [a_2, b_2]$ is understood as $[a_1 + a_2, b_1 + b_2]$.
- In this interpretation, the formula leads to the interval

$$[\alpha_1 \cdot 1 + \alpha_0 \cdot 0 + \alpha_{01} \cdot 0, \alpha_1 \cdot 1 + \alpha_0 \cdot 0 + \alpha_{01} \cdot 1] = [\alpha_1, \alpha_1 + \alpha_{01}].$$

11. Geometric analysis of the problem (cont-d)

- Thus:
  - to represent any interval $[a, b] \subseteq [0, 1]$ in this form,
  - it is sufficient to select the values α_i for which $\alpha_1 = a$, $\alpha_1 + \alpha_{01} = b$, and $\sigma_0 + \alpha_1 + \alpha_{01} = 1$.
- So, we can take $\alpha_1 = a$, $\alpha_{01} = b - a$ and, correspondingly,

$$\alpha_0 = 1 - (\alpha_1 + \alpha_{01}) = 1 - b.$$

12. In view of this geometric representation, how can we make it natural?

- In line with this geometric representation of interval-valued fuzzy techniques, how can make this natural?
- To look for such a natural representation, let us look for some natural phenomenon in which:
  - we have 3 basic states and
  - every other state can be represented as a convex combination of these basic states.
- Of course, there is such a phenomenon – it is the phenomenon of color:
  - there are three basic colors, e.g., red, green, and blue, and
  - every color can be represented as a convex combination of these colors.

13. In view of this geometric representation, how can we make it natural (cont-d)

- This is not just a theoretical idea:
  - this is how our eyes perceive color, via cells attuned to these three basic colors,
  - this is how computer screens represent color – by three dense grids of light sources corresponding to three basic colors, etc.
- So, we arrive at the following idea.

14. Main idea

- Let us represent each fuzzy interval $[a, b]$ by an appropriate color.
- Namely, by a convex combination of three basic colors with coefficients a, $b - a$ and $1 - b$.

15. Details

- Which of the three basic colors should correspond to the three basic elements 0, 1, and $[0, 1]$?
- Which basic colors should corresponding to 0, which to 1, and which to $[0, 1]$?
- From the commonsense viewpoint, the interval $[0, 1]$ is kind of between 0 ("absolutely false") and 1 ("absolutely true").
- Thus:
  - as a color corresponding to $[0, 1]$,
  - it is reasonable to select a color which is, in some reasonable sense, intermediate between the two others.
- For colors, there is a natural linear order.

16. Details (cont-d)

- This order corresponds to the frequency of the corresponding light. In this order:
  - red color has the smallest frequency,
  - blue has the largest frequency, and
  - green is in between.
- Thus, it is reasonable to associate green with the interval $[0, 1]$.
- How do we interpret the remaining two colors: red and blue?
- One of them should correspond to 0, another to one.
- With street lights, we are accustomed to associating red with prohibition and negation.
- So, it is reasonable to associate red with 0 (false) and blue with 1 (true).

17. Comment

- There is another reason for such an association.
- Indeed, "true" usually carries more information than "false".
- E.g., when we try to explain a natural phenomenon:
  - most explanation attempts turns our top be false, and
  - only one is true.
- From this viewpoint, it is reasonable:
  - to associate 1 (true)
  - with the color that carries more information.
- From this viewpoint:
  - since each cycle can convey a certain amount of information,
  - the information-carrying capacity of each color is proportional to its frequency.

18. Comment (cont-d)

- Since the blue color corresponds to higher frequency, it is more informative.
- Thus, it should be associated with 1.

19. Resulting color representation of interval-valued fuzzy degrees

- So, we arrive at the following representation of an interval $[a, b] \subseteq [0, 1]$: it is a convex combination, in which we have:
  - blue with coefficient b,
  - green with coefficient $b - a$, and
  - red with coefficient a.
- Our preliminary tests show that this scheme indeed provides users with a reasonable representation of uncertainty.

20. Color representation can be used not only to represent fuzzy degrees, but also to process them

- Up to now, we explained that colors provide a natural *representation* of interval-valued fuzzy degrees.
- Interestingly, the color representation can be also used for *processing* fuzzy degrees.
- Namely, we can process the corresponding colored beams of light.
- The main advantage of such optical data processing – and optical data processing in general – is that:
  - it can be naturally parallelized,
  - this drastically speeds up computations.
- An additional speed-up comes from the fact that:
  - optical computing occurs with the speed of light,
  - which is, according to modern physics, the fastest possible process in the Universe.

21. Color representation beyond interval-valued fuzzy degrees

- In describing colors, we may be interested not only in the color itself, but also in how bright or how light is this color.
- Interestingly, there is a natural way to extend our color-based fuzzy interpretation to this additional feature.
- Namely, in 2011, Lotfi Zadeh proposed to make our description of the expert's opinion more adequate by adding,
  - to the expert's degree of confidence – as described by a number from the interval $[0, 1]$ or by a subinterval of this interval,
  - an additional number that describes the expert's degree of confidence in this number.
- Zadeh called the resulting pairs *Z-numbers*.

22. Beyond interval-valued fuzzy degrees (cont-d)

- For example, when asked whether a person who one saw for a second was tall:
  - the witness may say "somewhat tall", and
  - explain that he/she is somewhat – but not fully – confident about this.
- Thus, in this case:
  - in addition to the numerical value corresponding to "somewhat tall",
  - we have an additional number describing "somewhat confident".
- This additional number describe the degree to which the expert is confident about his/her statement.

23. Beyond interval-valued fuzzy degrees (cont-d)

- This number is thus a natural counterpart of the intensity of color.
- Thus, a color scheme can be naturally used:
  - not only to describe interval-valued fuzzy degrees,
  - but also to describe interval-valued Z-numbers.

24. An additional confirmation that interval-valued fuzzy degrees are more adequate

- An interesting side effect of our idea is that we now have a new argument of:
  - why interval-valued fuzzy degrees are more adequate
  - than values from the interval $[0, 1]$ – at least for describing sufficiently complex phenomena.
- Indeed, as we have mentioned:
  - while interval-valued degrees can be obtained as convex combinations of 3 basic degrees, and thus, correspond to combinations of 3 colors,
  - $[0, 1]$-based degrees are convex combinations of 2 basic degrees, and thus, correspond to combinations of 2 colors.

25. An additional confirmation that interval-valued fuzzy degrees are more adequate (cont-d)

- Herein lies the crucial difference:
  - for general graphs, the question of whether a graph can be colored in 3 colors in known to be NP-hard,
  - while the question of whether a graph can be colored in 2 color can be answered by a simple feasible algorithm.
- Thus, if we want to be able to describe complex phenomena – for many of which important questions are NP-hard:
  - we cannot use 2 colors, that would lead to models which are too simple,
  - we need at least 3 colors.
- (And, by the way, 3 colors are sufficient to reflect the corresponding complexity.)

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