Logical Inference Inevitably Appears: Fuzzy-Based Explanation

Julio Urenda\textsuperscript{1}, Vladik Kreinovich\textsuperscript{2}, Olga Kosheleva\textsuperscript{3}, and Orsolya Csiszár\textsuperscript{4}

\textsuperscript{1,2}Departments of \textsuperscript{1}Mathematics, \textsuperscript{2}Computer Science and \textsuperscript{3}Teacher Education
University of Texas at El Paso, El Paso, Texas 79968, USA
jcurenda@utep.edu, vladik@utep.edu, olgak@utep.edu
Aalen University of Applied Sciences, Germany, orsolya.csiszar@hs-aalen.de
1. Main question: logical inference historically appeared, but was it inevitable?

- Many thousands years ago, our primitive ancestors did not have the ability to reason logically and to perform logical inference.
- This ability appeared later.
- A natural question is:
  - was this appearance inevitable,
  - or was this a lucky incident that could have been missed?
- In this talk, we use fuzzy techniques to provide a possible answer to this question.
- Our answer is: yes, the appearance of logical inference is inevitable.
2. Let us formulate this question in precise terms

- Nowadays, we know the statements which are absolutely true, namely, the statements of abstract mathematics.
- However, these statements already presuppose the ability to reason logically.
- We are interested in analyzing how logical reasoning appeared in the first place.
- So, we need to ignore mathematical statements and concentrate on statements about the real world.
- In this case:
  - if we go beyond observed facts – which are, of course, clearly true.
  - such statements always come with some degree of certainty.
3. Let us formulate this question in precise terms (cont-d)

- Indeed, we may observe some phenomenon many times, but it does not mean that we are 100% sure that this will always be true:
  - Every day, we see the sun rising in the morning, but one day, there is a solstice, and the sun is not visible.
  - Every day eating a certain plant is OK, but one day, a fungus attacks this plant, making it poisonous for humans, etc.

- So, we need to deal with statements that have some degree of uncertainty.
4. We can combine these statements into complex ones

- Once we have statements $S_1, S_2, \ldots$, we can combine them into logical combinations.
- For example, we can consider statements $S_1 \& S_2$, $S_3 \lor \neg S_4$, etc.
- One of the main ideas behind fuzzy logic is that:
  - if we know the degrees of certainty $d_i$ in statements $S_i$, then we can estimate our degree of certainty in a combined statement
  - by using the corresponding “and”-, “or”-, and “not”-operations $f_\&(a, b)$, $f_\lor(a, b)$, and $f_-(a, b)$.
- For historical reasons:
  - “and”-operations are usually known as $t$-norms, while
  - “or”-operations are usually known as $t$-conorms.
5. We can combine these statements into complex ones

- Let us consider the set $D$ of degrees of certainty of all possible combined statements.

- This set must be closed under these operations, i.e.,
  - if $a \in D$ and $b \in D$,
  - then we must have $f_\& (a, b) \in D$, $f_\lor (a, b) \in D$, and $f_\neg (a) \in D$. 

6. Let us restrict ourselves to intuitively reasonable “and”-operation

- For non-mathematical statements, a combined statement “$A$ and $B$” is, in general, stronger than each of the two statements $A$ and $B$.
- So, it makes sense to consider “and”-operations that are consistent with this intuitive idea, i.e., for which:
  - wherever $a < 1$ and $b < 1$,
  - we have $f_{\&}(a, b) < a$ and $f_{\&}(a, b) < b$. 
7. A person – or even a group – rarely deals with all possible degrees of certainty

- Even now, it is rare that the same group of people deal with statements of all kinds degree of certainty.
- For example:
  - mathematicians usually deal only with absolutely correct statements,
  - physicists usually deal with statements that are correct on the physical level – i.e., have some uncertainty in them,
  - biologists usually deal with statement that have even less degree of certainty,
  - philosophers – unless they follow a formal approach – usually deal with statement with even less certainty, etc.
- At each moment of time, there are several such groups of people.
Let us denote the number of such groups by \( n \).

Let us denote by \( D_1, \ldots, D_n \) the sets of degrees of certainty corresponding to each of these groups.
9. What does appearance of logical inference mean in these terms

- In general, logical inference means that the same person – or at least the same group of people– deals both:
  - with some statements, e.g., $S_1$ and $S_2$, and
  - with their logical combination, e.g., $S_1 \& S_2$.

- In these terms, the appearance of logical inference means that:
  - on some level,
  - some logical combination of statement from this level also belongs to this same level.

- Now, we are ready to formulate our result in precise terms.
10. Comment

- We want to maintain the greatest possible degree of generality.
- So, we will use the weakest possible assumptions – as long as we can get a proof.
- For example:
  - we will not assume that the degrees of certainty are numbers from the interval $[0, 1]$;
  - for example, we allow interval-values degrees of certainty, and
  - we will not assume that the “and”-operation is commutative.
11. Definitions and the main result

- By *logical development*, we mean the tuple \( \langle D, f_\& , f_\lor , f_\neg , D_1, \ldots , D_n \rangle \), where:
  - \( D \) is a partially ordered set that contains the largest element 1 and also contains at least one element different from 1;
  - elements of the set \( D \) will be called *degrees of certainty*;
  - \( f_\& : D \times D \to D \) is an associative operation on \( D \) for which \( f_\&(a, b) < a \) and \( f_\&(a, b) < b \) whenever \( a < 1 \) and \( b < 1 \);
  - \( f_\lor : D \times D \to D \) and \( f_\neg : D \to D \) are operations on \( D \); and
  - \( D_i \) are subsets of \( D \) for which \( \cup D_i = D \).

- We say that a value \( d \in D \) is a *logical combination* of the values \( d_1, \ldots , d_m \in D \) if \( d \) can be obtained from \( d_i \) by using operations \( f_\&(a, b), \ f_\lor(a, b), \) and \( f_\neg(a, b) \).

- For example, we may have \( d = f_\&(d_1, d_2) \), or \( d = f_\lor(d_3, f_\neg(d_4)) \), etc.
12. Definitions and the main result (cont-d)

- We say that a logical development contains logical reasoning if one of the sets $D_i$ contains both:
  
  - some values $d_1, \ldots, d_m$, and  
  - a value $d$ which is their logical combination.

- **Proposition.** Every logical development contains logical reasoning.
13. Discussion

- This result means that:
  - as we consider more and more statements, eventually, there will be the case
  - when some group will be dealing both with some statements \textit{and} with their logical combination.

- In other words, logical inference will indeed inevitably appear.

- The above proposition promised the existence of \textit{some} logical combination.

- We will actually prove a more specific result: that on every logical development, there is a group $D_i$ that contains both:
  - some elements $d$ and $d'$, and
  - their “and”-combination $f_{\&}(d, d')$. 
14. Proof

- By definition, the set $D$ contains a degree $d_1$ which is smaller than 1.
- Let us consider, for each natural number $k > 1$, the degree $d_k$ that is obtained by applying $k$ times the “and”-operation $f_\&$ to $d_1$:
  \[
  d_2 = f_\&(d_1, d_1), \quad d_3 = f_\&(d_2, d_1) = f_\&(f_\&(d_1, d_1), d_1),
  \]
  \[
  d_4 = f_\&(d_3, d_1) = f_\&(f_\&(f_\&(d_1, d_1), d_1), d_1),
  \]
  and, in general, $d_{k+1} = f_\&(d_k, d_1)$.
- By associativity, we can conclude that for all possible value $k$ and $\ell$, we have $f_\&(d_k, d_\ell) = d_{k+\ell}$.
- We have $f_\&(a, b) < a$ and $f_\&(a, b) < b$ whenever $a < 1$ and $b < 1$.
- So, we can prove, by induction, that the degrees $d_k$ form a strictly decreasing sequence:
  \[
  1 > d_1 > d_2 > \ldots > d_k > d_{k+1} > \ldots
  \]
- This implies, in particular, that all the values $d_k$ are different.
15. Proof (cont-d)

- Since $\bigcup D_i = D$, for each $k$, the degree $d_k$ belongs to one of the groups $D_i$.

- Let $N_i$ denote the set of all the indices $k$ for which $d_k \in D_i$.

- Then, we have $N = \bigcup N_i$.

- Now, we can use Schur’s theorem, according to which:
  - every time we divide the set of all natural numbers into finitely many subsets $N_i$,
  - one of these subsets – let us denote it by $N_j$ – contains integers $k$ and $\ell$ for which the sum $k + \ell$ is also contained in $N_j$.

- Strictly speaking, Schur’s theorem requires that we have a partition.

- The sets $N_i$ do not necessarily form a partition – some of them may have a non-empty intersection.
16. Proof (cont-d)

- However, this problem is easy to overcome if:
  - instead of the original sets $N_1, N_2, \text{etc.}$,
  - we consider sets $N'_1 = N_1$, $N'_2 = N_2 - N_1$,
    \[ N'_3 = N_3 - (N_1 \cup N_2), \text{ and, in general,} \]
    \[ N'_i = N_i - (N_1 \cup \ldots \cup N_{i-1}). \]
- Then, the sets $N'_i$ form a partition.
- Thus, by Schur’s Theorem, there exists a set $N'_j$ that contains two numbers $k, \ell$, and their sum $k + \ell$.
- Since $N'_j \subseteq N_j$, the original set $N_j$ also contains these three numbers.
- By definition of the sets $N_j$, the fact that $k, \ell$, and $k + \ell$ all belong to $N_j$ means that $d_k \in D_j$, $d_\ell \in N_j$, and $d_{k+\ell} \in D_j$.
- This implies that $f_{&}(d_k, d_\ell) \in D_j$.
- The proposition is thus proven.
17. Discussion

- The above proposition says that for every $n$:
  - if we continuously add degree of certainty so that eventually all degrees will be added,
  - then, at some stage, we will reach a point at which logical reasoning emerges.

- In this result, the point at which logical reasoning emerges may depend on the specific division of the set $D$ into groups.

- However, there exists a stronger version of Schur’s theorem according to which, for each $n$, there exists a number $N(n)$ for which:
  - if we divide all the natural numbers from 1 to $N(n)$ into $n$ groups $N_1, \ldots, N_n$,  
  - then one of these groups $N_j$ contains some values $k$ and $\ell$ for which $k + \ell \in N_j$.  

18. Discussion (cont-d)

• In our terms, this means that:
  
  – if we only consider degrees $d_1, \ldots, d_{N(n)}$,
  
  – then among these degrees, one of the groups $D_j$ will contain elements $d_k, d_\ell$, and $d_{k+\ell} = f_\& (d_k, d_\ell)$.
19. A slightly stronger result

- Another generalization of the original Schur’s theorem is Folkman’s theorem, according to which:
  - for each division of the set of natural numbers $N$ into a finite number of subsets $N_i$, and for each $m > 1$,
  - there exists a subset $N_j$ and $m$ elements from this subset for which the sum of any number of them is still in $N_j$.

- In our terms, this means that:
  - not only we have two degrees $d_k, d_\ell \in D_j$ for which $f_\& (d_k, d_\ell) \in D_j$, but
  - we also have $m$ elements $d_{k_1}, \ldots, d_{k_m} \in D_j$ for which any “and”-combination $f_\& (d_{k_{j(1)}}, d_{k_{j(2)}}, \ldots)$ also belongs to $D_j$. 
20. A slightly stronger result (cont-d)

- In other words:
  - not only the simplest form of logical inference eventually appear, but also
  - more and more sophisticated versions of logical reasoning eventually appear.
21. Acknowledgments

This work was supported in part by:

- National Science Foundation grants 1623190, HRD-1834620, HRD-2034030, and EAR-2225395;

- AT&T Fellowship in Information Technology;

- program of the development of the Scientific-Educational Mathematical Center of Volga Federal District No. 075-02-2020-1478, and

- a grant from the Hungarian National Research, Development and Innovation Office (NRDI).
22. References about Schur’s theorem


