Logical Inference Inevitably Appears: Fuzzy-Based Explanation

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1. Main question: logical inference historically appeared, but was it inevitable?

- Many thousands years ago, our primitive ancestors did not have the ability to reason logically and to perform logical inference.
- This ability appeared later.
- A natural question is:
  - was this appearance inevitable,
  - or was this a lucky incident that could have been missed?
- In this talk, we use fuzzy techniques to provide a possible answer to this question.
- Our answer is: yes, the appearance of logical inference in inevitable.

2. Let us formulate this question in precise terms

- Nowadays, we know the statements which are absolutely true, namely, the statements of abstract mathematics.
- However, these statement already presuppose the ability to reason logically.
- We are interested in analyzing how logical reasoning appeared in the first place.
- So, we need to ignore mathematical statements and concentrate on statements about the real world.
- In this case:
  - if we go beyond observed facts – which are, of course, clearly true.
  - such statements always come with some degree of certainty.

3. Let us formulate this question in precise terms (cont-d)

- Indeed, we may observe some phenomenon many times, but it does not mean that we are 100% sure that this will always be true:
  - Every day, we see the sun rising in the morning, but one day, there is a solstice, and the sun is not visible.
  - Every day eating a certain plant is OK, but one day, a fungus attacks this plant, making it poisonous for humans, etc.
- So, we need to deal with statements that have some degree of uncertainty.

4. We can combine these statements into complex ones

- Once we have statements S_1, S_2, $\ldots$, we can combine them into logical combinations.
- For example, we can consider statements $S_1 \,\&\, S_2$, $S_3 \vee \neg S_4$, etc.
- One of the main ideas behind fuzzy logic is that:
  - if we know the degrees of certainty d_i in statements S_i, then we can estimate our degree of certainty in a combined statement
  - by using the corresponding "and"-, "or"-, and "not"-operations $f_\&(a, b)$, $f_\vee(a, b)$, and $f_\neg(a, b)$.
- For historical reasons:
  - "and"-operations are usually known as *t-norms*, while
  - "or"-operations are usually known as *t-conorms*.

5. We can combine these statements into complex ones

- Let us consider the set D of degrees of certainty of all possible combined statements.
- This set must be closed under these operations, i.e.,
  - if $a \in D$ and $b \in D$,
  - then we must have $f_\&(a, b) \in D$, $f_\vee(a, b) \in D$, and $f_\neg(a) \in D$.

6. Let us restrict ourselves to intuitively reasonable "and"-operation

- For non-mathematical statements, a combined statement "A and B" is, in general, stronger than each of the two statements A and B.
- So, it makes sense to consider "and"-operations that are consistent with this intuitive idea, i.e., for which:
  - wherever $a < 1$ and $b < 1$,
  - we have $f_\&(a, b) < a$ and $f_\&(a, b) < b$.

7. A person – or even a group – rarely deals with all possible degrees of certainty

- Even now, it is rare that the same group of people deal with statements of all kinds degree of certainty.
- For example:
  - mathematicians usually deal only with absolutely correct statements,
  - physicists usually deal with statements that are correct on the physical level – i.e., have some uncertainty in them,
  - biologists usually deal with statement that have even less degree of certainty,
  - philosophers – unless they follow a formal approach – usually deal with statement with even less certainty, etc.
- At each moment of time, there are several such groups of people.

8. A person – or even a group – rarely deals with all possible degrees of certainty (cont-d)

- Let us denote the number of such groups by n.
- Let us denote by $D_1, \ldots, D_n$ the sets of degrees of certainty corresponding to each of these groups.

9. What does appearance of logical inference mean in these terms

- In general, logical inference means that the same person – or at least the same group of people– deals both:
  - with some statements, e.g., S_1 and S_2, and
  - with their logical combination, e.g., $S_1 \,\&\, S_2$.
- In these terms, the appearance of logical inference means that:
  - on some level,
  - some logical combination of statement from this level also belongs to this same level.
- Now, we are ready to formulate our result in precise terms.

10. Comment

- We want to maintain the greatest possible degree of generality.
- So, we will use the weakest possible assumptions – as long as we can get a proof.
- For example:
  - we will not assume that the degrees of certainty are numbers from the interval $[0, 1]$;
  - for example, we allow interval-values degrees of certainty, and
  - we will not assume that the "and"-operation is commutative.

11. Definitions and the main result

- By *logical development*, we mean the tuple $\langle D, f_\&, f_\vee, f_\neg, D_1, \ldots, D_n\rangle$, where:
  - D is a partially ordered set that contains the largest element 1 and also contains at least one element different from 1;
  - elements of the set D will be called *degrees of certainty*;
  - $f_\& : D \times D \to D$ is an associative operation on D for which $f_\&(a, b) < a$ and $f_\&(a, b) < b$ whenever $a < 1$ and $b < 1$;
  - $f_\vee : D \times D \to D$ and $f_\neg : D \to D$ are operations on D; and
  - D_i are subsets of D for which $\cup D_i = D$.
- We say that a value $d \in D$ is a *logical combination* of the values $d_1, \ldots, d_m \in D$ if d can be obtained from d_i by using operations

$$f_\&(a, b), \quad f_\vee(a, b), \text{ and } f_\neg(a, b).$$

- For example, we may have $d = f_\&(d_1, d_2)$, or $d = f_\vee(d_3, f_\neg(d_4))$, etc.

12. Definitions and the main result (cont-d)

- We say that a logical development *contains logical reasoning* if one of the sets D_i contains both:
  - some values $d_1, \ldots, d_m$, and
  - a value d which is their logical combination.
- **Proposition.** *Every logical development contains logical reasoning.*

13. Discussion

- This result means that:
  - as we consider more and more statements, eventually, there will be the case
  - when some group will be dealing both with some statements *and* with their logical combination.
- In other words, logical inference will indeed inevitably appear.
- The above proposition promised the existence of *some* logical combination.
- We will actually prove a more specific result: that on every logical development, there is a group D_i that contains both:
  - some elements d and d', and
  - their "and"-combination $f_{\&}(d, d')$.

14. Proof

- By definition, the set D contains a degree d_1 which is smaller than 1.
- Let us consider, for each natural number $k > 1$, the degree d_k that is obtained by applying k times the "and"-operation $f_\&$ to d_1:

$$d_2 = f_\&(d_1, d_1), \quad d_3 = f_\&(d_2, d_1) = f_\&(f_\&(d_1, d_1), d_1),$$

$$d_4 = f_\&(d_3, d_1) = f_\&(f_\&(f_\&(d_1, d_1), d_1), d_1),$$

$$\text{and, in general, } d_{k+1} = f_\&(d_k, d_1).$$

- By associativity, we can conclude that for all possible value k and ℓ, we have $f_\&(d_k, d_\ell) = d_{k+\ell}$.
- We have $f_\&(a, b) < a$ and $f_\&(a, b) < b$ whenever $a < 1$ and $b < 1$.
- So, we can prove, by induction, that the degrees d_k form a strictly decreasing sequence:

$$1 > d_1 > d_2 > \ldots > d_k > d_{k+1} > \ldots$$

- This implies, in particular, that all the values d_k are different.

15. Proof (cont-d)

- Since $\cup D_i = D$, for each k, the degree d_k belongs to one of the groups D_i.
- Let N_i denote the set of all the indices k for which $d_k \in D_i$.
- Then, we have $N = \cup N_i$.
- Now, we can use Schur's theorem, according to which:
  - every time we divide the set of all natural numbers into finitely many subsets N_i,
  - one of these subsets – let us denote it by N_j – contains integers k and ℓ for which the sum $k + \ell$ is also contained in N_j.
- Strictly speaking, Schur's theorem requires that we have a partition.
- The sets N_i do not necessarily form a partition – some of them may have a non-empty intersection.

16. Proof (cont-d)

- However, this problem is easy to overcome if:
  - instead of the original sets N_1, N_2, etc.,
  - we consider sets $N'_1 = N_1$, $N'_2 = N_2 - N_1$,

$$N'_3 = N_3 - (N_1 \cup N_2),\text{ and, in general,}$$

$$N'_i = N_i - (N_1 \cup \ldots \cup N_{i-1}).$$

- Then, the sets N'_i form a partition.
- Thus, by Schur's Theorem, there exists a set N'_j that contains two numbers k, ℓ, and their sum $k + \ell$.
- Since $N'_j \subseteq N_j$, the original set N_j also contains these three numbers.
- By definition of the sets N_j, the fact that k, ℓ, and $k + \ell$ all belong to N_j means that $d_k \in D_j$, $d_\ell \in N_j$, and $d_{k+\ell} \in D_j$.
- This implies that $f_\&(d_k, d_\ell) \in D_j$.
- The proposition is thus proven.

17. Discussion

- The above proposition says that for every n:
  - if we continuously add degree of certainty so that eventually all degrees will be added,
  - then, at some stage, we will reach a point at which logical reasoning emerges.
- In this result, the point at which logical reasoning emerges may depend on the specific division of the set D into groups.
- However, there exists a stronger version of Schur's theorem according to which, for each n, there exists a number $N(n)$ for which:
  - if we divide all the natural numbers from 1 to $N(n)$ into n groups $N_1, \ldots, N_n$,
  - then one of these groups N_j contains some values k and ℓ for which $k + \ell \in N_j$.

18. Discussion (cont-d)

- In our terms, this means that:
  - if we only consider degrees $d_1, \ldots, d_{N(n)}$,
  - then among these degrees, one of the groups D_j will contain elements d_k, d_ℓ, and $d_{k+\ell} = f_\&(d_k, d_\ell)$.

19. A slightly stronger result

- Another generalization of the original Schur's theorem is Folkman's theorem, according to which:
  - for each division of the set of natural numbers N into a finite number of subsets N_i, and for each $m > 1$,
  - there exists a subset N_j and m elements from this subset for which the sum of any number of them is still in N_j.
- In our terms, this means that:
  - not only we have two degrees $d_k, d_\ell \in D_j$ for which $f_\&(d_k, d_\ell) \in D_j$, but
  - we also have m elements $d_{k_1}, \ldots, d_{k_m} \in D_j$ for which any "and"-combination $f_\&(d_{k_{j(1)}}, d_{k_{j(2)}}, \ldots)$ also belongs to D_j.

20. A slightly stronger result (cont-d)

- In other words:
  - not only the simplest form of logical inference eventually appear, but also
  - more and more sophisticated versions of logical reasoning eventually appear.

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