How to Propagate Interval (and Fuzzy) Uncertainty: Optimism-Pessimism Approach

Vinícius F. Wasques\textsuperscript{a}, Olga Kosheleva\textsuperscript{b}, and Vladik Kreinovich\textsuperscript{c}

\textsuperscript{a}Ilum School of Science, São Paulo State University UNESP
São Paulo, Brazil, vwasques@outlook.com

\textsuperscript{b,c}Departments of \textsuperscript{b}Teacher Education and \textsuperscript{c}Computer Science
University of Texas at El Paso, El Paso, Texas 79968, USA
olgak@utep.edu, vladik@utep.edu
1. Need for uncertainty propagation

- To predict the future value $y$ of one of the desired quantities:
  - we find the current quantities $x_1, \ldots, x_n$ that are related to $y$ by a known dependence $y = f(x_1, \ldots, x_n)$,
  - we get estimates $\tilde{x}_1, \ldots, \tilde{x}_n$ of the current quantities – coming either from measurements or from experts, and
  - we compute the estimate $\tilde{y} = f(\tilde{x}_1, \ldots, \tilde{x}_n)$ for the desired quantity $y$.

- Often, the only information that we have about the measurement error is the range $[\Delta, \bar{\Delta}]$ of its possible values of measurement error.

- Then, the only conclusion we can make about the actual (unknown) value $x$ is that $x$ is in $[x, \bar{x}] \overset{\text{def}}{=} [\tilde{x} - \bar{\Delta}, \tilde{x} - \Delta]$.

- This situation is known as *interval uncertainty*. 
2. Propagation of interval uncertainty: usual approach

- If all we know about each input $x_i$ is that its value is located somewhere in the interval $x_i = [x_i, \bar{x}_i]$, then:
  - the only thing we can conclude about the value $y = f(x_1, \ldots, x_n)$ is that this value is somewhere in the set
  $$y = f(x_1, \ldots, x_n) \overset{\text{def}}{=} \{ f(x_1, \ldots, x_n) : x_1 \in x_1, \ldots, x_n \in x_n \}.$$  
- The task of computing this set is known as *interval computations*.
- In this case, we have
  $$f([x_1, \bar{x}_1], \ldots, [x_n, \bar{x}_n]) = [f(x_1, \ldots, x_n), f(\bar{x}_1, \ldots, \bar{x}_n)].$$
- For example, for addition, we get $[x_1, \bar{x}_1] + [x_2, \bar{x}_2] = [x_1 + x_2, \bar{x}_1 + \bar{x}_2]$. 
3. Problem with the usual approach

• The usual approach corresponds to the most pessimistic case, when we consider the worst possible scenario.

• For example, in the case of addition, we consider the case when:
  – each input $x_i$ attain its largest possible value – i.e.,
  – the corresponding measurement error attains its smallest possible value.

• We have already mentioned that it is not very probable that the measurement error will attain its smallest possible value.

• That two such rare events will both happen is highly improbable.

• It is therefore desirable to come up with more realistic estimates of the results of data processing.

• This is what we will be doing in this talk.
4. Let us start with the optimistic approach

- We know that for each input $x_i$, both endpoints $\underline{x}_i$ and $\overline{x}_i$ are possible.
- If one or both of these values was not possible, we would have a narrower interval of possible values of $x_i$.
- We have no confidence about the possibility of intermediate values.
- So, to minimize the set of all possible values of $y = f(x_1, \ldots, x_n)$, let us consider the case when only these two values are possible for $x_i$.
- Then, the set $X$ of possible tuples $x = (x_1, \ldots, x_n)$ must have the property that for each $i$:
  - at least one tuple must contain $\underline{x}_i$, and
  - at least one tuple must contain $\overline{x}_i$.
- For each set with this property, we can consider the interval hulls generated by all the corresponding values $f(x)$.
- Then, we take the intersection $f_o(x_1, \ldots, x_n)$ of all these hulls.
5. Cases of $n = 2$ and of monotonic $f$

- **Proposition 1.** For $n = 2$, $f_o([\underline{x}_1, \overline{x}_1], [\underline{x}_2, \overline{x}_2]) = [\Delta, \overline{\Delta}]$, where
  \[
  \Delta = \max(\min(f(\underline{x}_1, \overline{x}_2), f(\overline{x}_1, \overline{x}_2)), \min(f(\underline{x}_1, \overline{x}_2), f(\overline{x}_1, \underline{x}_2)))
  \text{ and }
  \overline{\Delta} = \min(\max(f(\underline{x}_1, \overline{x}_2), f(\overline{x}_1, \overline{x}_2)), \max(f(\underline{x}_1, \overline{x}_2), f(\overline{x}_1, \underline{x}_2))).
  \]

- **Proposition 2.** If $f(\underline{x}_1, \overline{x}_2)$ be (non-strictly) increasing in each of its variables, $f_o([\underline{x}_1, \overline{x}_1], [\underline{x}_2, \overline{x}_2]) = [\Delta, \overline{\Delta}]$, where
  \[
  \Delta = \min(f(\underline{x}_1, \overline{x}_2), f(\overline{x}_1, \overline{x}_2)) \text{ and } \overline{\Delta} = \max(f(\underline{x}_1, \overline{x}_2), f(\overline{x}_1, \underline{x}_2)).
  \]

- This interval is equal to $f(X)$, for $X = \{(\underline{x}_1, \overline{x}_2), (\overline{x}_1, \underline{x}_2)\}$.

- Since addition is clearly increasing in each of the variables, we have
  \[
  [\underline{x}_1, \overline{x}_1] \circ [\underline{x}_1, \overline{x}_2] = [\min(\underline{x}_1 + \overline{x}_2, \overline{x}_1 + \underline{x}_2), \max(\underline{x}_1 + \overline{x}_2, \overline{x}_1 + \underline{x}_2)].
  \]

- These operations are known as *Kaucher arithmetic*. 
6. Case of general $n$

- What happened with addition of intervals for the generic $n$ – which corresponds to the linearization case of uncertainty propagation?

- In this case, we have the following result.

- By the *midpoint* of an interval $[x_i, \bar{x}_i]$, we mean $m_i = (x_i + \bar{x}_i)/2$.

- By the *half-width* (radius) of an interval $[x_i, \bar{x}_i]$, we mean the value $r_i = (\bar{x}_i - x_i)/2$.

- **Proposition 3.** For addition $f(x_1, \ldots, x_n) = x_1 + \ldots + x_n$ of intervals $[x_i, \bar{x}_i]$, the optimistic range has the following form:
  
  - If $2 \max_i r_i \geq \sum_j r_j$, then $\sum_o [x_i, \bar{x}_i] = [m - r, m + r]$, where $m = m_1 + \ldots + m_n$ and $r = 2 \max_i r_i - \sum_j r_j$.

  - If $2 \max_i r_i < \sum_j r_j$, then $\sum_o [x_i, \bar{x}_i] = \emptyset$. 
7. Limitations of the optimistic approach

- For two intervals, optimistic approach means that:
  - when \( x_1 \) attains its largest possible value \( \bar{x}_1 \),
  - the quantity \( x_2 \) attains its smallest possible value \( \underline{x}_2 \).

- This is as improbable as the corresponding assumption in the pessimistic case.

- Thus, both pessimistic and optimistic approaches are not realistic.

- The pessimistic approach is too pessimistic, it consider too many possible values of \( \Delta y \).

- The optimistic approach is too optimistic, it consider too few possible values of \( \Delta y \).

- So, to get a realistic estimate, we need to combine the pessimistic and optimistic results into a single optimistic-pessimistic approach.
8. How to combine pessimistic and optimistic intervals

- Let us first consider the case when the optimistic approach leads to a non-empty interval.
- In this case, we have two intervals: the pessimistic interval $A$ and the optimistic intervals $B$.
- We need to combine them into a single interval.
- This problem is similar to the general problem that we started with:
  - if we knew how to combine two numerical estimates $a$ and $b$ into a single numerical estimate $c(a, b)$,
  - then a reasonable idea would be to consider the set of all possible values $c(a, b)$ when $a \in A$ and $b \in B$:
    \[ c(A, B) = \{ c(a, b) : a \in A, b \in B \}. \]
- To utilize this idea, we need to find reasonable ways to combine two numerical estimates.
9. How to combine two numerical estimates

- What are the reasonable properties of the combination function $c(a, b)$?
- We have two numerical estimates of the same quantity.
- So, a reasonable requirement is that the combined estimate should be within the interval generated by these two estimates:
  \[
  \min(a, b) \leq c(a, b) \leq \max(a, b).
  \]
- The second reasonable property comes from the fact that often, what we estimate is a combination of two or more parts.
- E.g., we may be estimating the population of a state, which is equal to the sum of populations of all its counties.
- In particular, for the case of two parts, we have estimates $a_1$ and $b_1$ for the first part, and estimates $a_2$ and $b_2$ for the second part.
10. How to combine two numerical estimates (cont-d)

- We can:
  - first combine estimates for each part, into two combined estimates $c(a_1, b_1)$ and $c(a_2, b_2)$, and
  - then add the resulting combined estimates into a single value $c(a_1, b_1) + c(a_2, b_2)$.

- Alternatively, we can first add the estimates for both parts, resulting in $a = a_1 + a_2$ and $b = b_1 + b_2$, and then combine these two estimates.

- This results in $c(a, b) = c(a_1 + a_2, b_1 + b_2)$.

- It is reasonable to require that these two ways lead to the same value, i.e., that $c(a_1, b_1) + c(a_2, b_2) = c(a_1 + a_2, b_1 + b_2)$.

- We say that a function $c(a, b)$ is a combination rule if it satisfies the above conditions.
11. How to combine two numerical estimates (cont-d)

- **Proposition 4.** A function is a combination rule if and only if it has the form \( c(a, b) = \gamma \cdot a + (1 - \gamma) \cdot b \) for some \( \gamma \in [0, 1] \).

- The above function is non-strictly increasing in both \( a \) and \( b \).

- Thus we can come up with an explicit expression for the combined interval: \( c([a, \bar{a}], [b, \bar{b}]) = [\gamma \cdot a + (1 - \gamma) \cdot b, \gamma \cdot \bar{a} + (1 - \gamma) \cdot \bar{b}] \).

- Interestingly:
  - if we take the pessimistic interval as \([a, \bar{a}]\) and the optimistic interval as \([b, \bar{b}]\),
  - we get the formulas for so-called *interactive addition for intervals*.

- These formulas that have been successfully used in many applications.

- Thus, our analysis provides a new justification for these formulas.
12. What If the Optimistic Approach Leads to the Empty Set

- But what can we do if the optimistic approach leads to the empty set?
- In this case, we only have one interval – the pessimistic one.
- So, to get a narrower realistic interval, we can only use this given interval.
- In other words, we need an operation $F$ that takes an interval $[a, \bar{a}]$ and returns its subinterval $F([a, \bar{a}]) \subseteq [a, \bar{a}]$.
- To analyze what are the natural properties of this transformation, let us take into account that:
  - we are processing numerical values of the corresponding quantities, but
  - these numerical values depends on the selection of the measuring unit, on the selection of the starting point.
- Sometimes, they also depend on the choice of the sign.
13. Empty Set Case (cont-d)

- For example, what is positive electric charge and what is negative is just a question of choice.

- If we change the measuring unit to a new one which is $\lambda$ times smaller, then all numerical values are multiplies by $\lambda$: $x \mapsto \lambda \cdot x$.

- For example, 2 meters becomes 200 centimeters.

- If we change the direction, then all values are multiplied by $-1$: $x \mapsto -x$.

- If we change the starting point to a new one which is $x_0$ units earlier, then we need to add $x_0$ to all numerical values: $x \mapsto x + x_0$.

- For example, Year $x = 2$ in the French revolution calendar – that started in 1789 – is year $2 + 1789 = 1791$ in the regular calendar.

- In general, if we make all these changes, we get a linear transformation $x \mapsto \lambda \cdot x + x_0$ for all possible real numbers $\lambda \neq 0$ and $x_0$. 
14. Empty Set Case (cont-d)

- It is reasonable to require that the selection of a subinterval should not depend on these choices.

- We say that a function $F$ that maps each interval into its subinterval is *scale-invariant* if:
  
  - for every interval $[a, \overline{a}]$ and for all possible real numbers $\lambda \neq 0$ and $x_0$,
  - we have the following property:

    $$\text{if } F([a, \overline{a}]) = [b, \overline{b}], \text{ then we should have } F([A, \overline{A}]) = [B, \overline{B}],$$

    where $[A, \overline{A}] \overset{\text{def}}{=} \lambda \cdot [a, \overline{a}] + x_0$ and $[B, \overline{B}] \overset{\text{def}}{=} \lambda \cdot [b, \overline{b}] + x_0$.

- **Proposition 5.** A function $F$ that maps each interval into its subinterval:
  
  - is scale-invariant if and only if
  - it has the form $F([m - r, m + r]) = [m - \gamma \cdot r, m + \gamma \cdot r]$ for some $\gamma \in [0, 1]$. 