Random Interval Arithmetic is Closer to Common Sense: An Observation

René Alt and Jean-Luc Lamotte
Laboratoire d'Informatique de Paris 6
Université Pierre et Marie Curie
4 Place Jussieu, 75252 cedex 05, Paris, France
rene.alt@upmc.fr, Jean-Luc.Lamotte@lip6.fr

Vladik Kreinovich
Department of Computer Science
University of Texas, El Paso, TX 79968
email vladik@cs.utep.edu

1. Commonsense Arithmetic

- We have a bridge whose weight we know with an accuracy of 1 ton.
- On this bridge, we place a car whose weight we know with an accuracy of 5 kg.
- The accuracy of the overall weight is still 1 ton.
- This is what an engineer or a physicist would say.
- *Related joke:*
  - in 2000, a dinosaur was 14,000,000 years old;
  - so, in 2005, it must be 14,000,005 years old.
- *What is desired:* if $\Delta_a \gg \Delta_b$, and
  - we add "uncertainty approximately Δ_b" to "uncertainty approximately Δ_a",
  - we should get "uncertainty approximately Δ_a".

2. Traditional Interval Arithmetic Does not Have the Desired Property

- A natural way of dealing with approximately known values is *interval arithmetic.*
- The value $\widetilde{a}$ with an accuracy Δ_a is interpreted as an interval $[\widetilde{a}-\Delta_a, \widetilde{a}+\Delta_a]$.
- *Specifics:*
  - we know $\widetilde{a}$ with uncertainty Δ_a;
  - we know $\widetilde{b}$ with uncertainty Δ_b;
  - then, $\mathbf{a} = [\widetilde{a} - \Delta_a, \widetilde{a} + \Delta_a]$, $\mathbf{b} = [\widetilde{b} - \Delta_b, \widetilde{b} + \Delta_b]$, and
  - so, the set of possible values of $c = a + b$ is an interval

$$\mathbf{c} = \mathbf{a} + \mathbf{b} = [(\widetilde{a} + \widetilde{b}) - (\Delta_a + \Delta_b), (\widetilde{a} + \widetilde{b}) + (\Delta_a + \Delta_b)].$$

3. Traditional Interval Arithmetic Does not Have the Desired Property

- *Situation:*
  - we know $\widetilde{a}$ with uncertainty Δ_a;
  - we know $\widetilde{b}$ with uncertainty Δ_b;
  - we conclude that

$$c \in [(\widetilde{a} + \widetilde{b}) - (\Delta_a + \Delta_b), (\widetilde{a} + \widetilde{b}) + (\Delta_a + \Delta_b)].$$

- *Interpretation:* we thus interpret this interval as

$$\text{“}\widetilde{a} + \widetilde{b} \text{ with uncertainty } \Delta_a + \Delta_b\text{”}.$$

- *Conclusion:* if we know a with uncertainty 1 ton, and we know b with uncertainty 5 kg, then the resulting uncertainty in $a + b$ is 1.005 ton.
- *Problem:* how can we modify interval arithmetic?

4. Interval Arithmetic: Origins

- *Objective:* analyze how:
  - the uncertainty in input data, and
  - the round-off imprecision of computer operations

  affect the results of the computations.
- *Traditional approach:* statistical techniques.
- *Problem:*
  - we must know the exact probability distributions of the input and round-off errors;
  - in practice, we *don't know* these distributions.
- *What we do know:* upper bounds on the errors – i.e., *intervals* that contain them.
- *e.g.:* space navigation under uncertainty (NASA, 1950s).
- *Interval arithmetic* was developed.

5. Interval Arithmetic: Limitations

- *Problem:* producing the exact bounds on the inaccuracy of the output is often difficult (NP-hard).
- *Discussion:* the origin of interval techniques is in NASA-related problems that required high reliability.
- *Conclusion:* the emphasis in interval computations has always been on getting the *validated* results.
- Interval techniques produce estimates that are guaranteed to contain (enclose) the actual error.
- *Limitation:* it is often desirable,
  - in addition to guaranteed "overestimates",
  - to produce a reasonable estimate of the size of the actual error,
  - an estimate that may be only valid with a certain probability.

6. Interval Arithmetic: Main Idea

- *Main idea:* we follow computations step by step.
- *Specifics:* for each intermediate computation step $z := x \odot y$,
  - once we have already computed the intervals $\mathbf{x} = [\underline{x}, \overline{x}]$ and $\mathbf{y} = [\underline{y}, \overline{y}]$ of possible values of x and y,
  - we compute the interval for z.
- *Traditional interval arithmetic:* apply interval arithmetic operation to $\mathbf{x}$ and $\mathbf{y}$ corresponding to the worst case.
- *Example:* for addition,

$$\mathbf{z} = [\underline{x} + \underline{y}, \overline{x} + \overline{y}].$$

7. Random Interval Arithmetic

- *New idea (Vignes et al.) – motivation:*
  - depending on the relative monotonicity of the x and y relative to inputs,
  - the intervals $\mathbf{z}$ can change from the worst-case situation to the best-case situation.
- *Best case arithmetic:* (a.k.a. dual or inner): e.g., for addition,

$$\mathbf{z} = [\min(\underline{x} + \overline{y}, \overline{x} + \underline{y}), \max(\underline{x} + \overline{y}, \overline{x} + \underline{y})].$$

- *Reasonable assumption:*
  - monotonicity in the same direction and
  - monotonicity in different directions

  are equally frequent.
- *Result:* on each step, we pick traditional or inner arithmetic with equal probability.

8. Random Interval Arithmetic Has the Desired Property

- *Example:* addition $\mathbf{c} = \mathbf{a} + \mathbf{b}$.
- *Traditional arithmetic:* the half-width is:

$$\Delta_c^t = \Delta_a + \Delta_b.$$

- *Dual arithmetic:* $\Delta_c^d = \max(\Delta_a, \Delta_b) - \min(\Delta_a, \Delta_b)$.
- *Random interval arithmetic:* uses each operation with probability 50%.
- So, the average half-width of $\mathbf{c}$ is $\Delta_c^r = \dfrac{\Delta_c^t + \Delta_c^d}{2}$.
- *Fact:* $\Delta_c^t = \Delta_a + \Delta_b = \max(\Delta_a, \Delta_b) + \min(\Delta_a, \Delta_b)$.
- *Conclusion:* $\Delta_c^r = \max(\Delta_a, \Delta_b)$.
- *Good news:* this is exactly the intuitive property that we have been trying to formalize.

9. What If We Add n Values?

- *Problem:*
  - we know each quantity a_i with an accuracy Δ_i;
  - what is the (expected value of) the accuracy in $a = a_1 + \ldots + a_n$?
- First, we add $a_1 + a_2$; the resulting accuracy is $\max(\Delta_1, \Delta_2)$.
- To estimate the uncertainty of the next intermediate result $(a_1 + a_2) + a_3$, we take, as an estimate of the uncertainty in $a_1 + a_2$, the value $\max(\Delta_1, \Delta_2)$.
- Then, the average uncertainty in $(a_1 + a_2) + a_3$ will be equal to

$$\max(\max(\Delta_1, \Delta_2), \Delta_3) = \max(\Delta_1, \Delta_2, \Delta_3).$$

- Similarly, we conclude that the average uncertainty in $a_1 + \ldots + a_n$ is equal to

$$\max(\Delta_1, \ldots, \Delta_n).$$

10. Computing $f(a_1, a_2)$

- When $\Delta a_i \stackrel{\text{def}}{=} a_i - \widetilde{a}_i \ll a_i$, we can safely linearize the expression for $f(a_1, a_2)$:

$$f(a_1, a_2) = f(\widetilde{a}_1 + \Delta a_1, \widetilde{a}_2 + \Delta a_2) =$$

$$f(\widetilde{a}_1, \widetilde{a}_2) + \frac{\partial f}{\partial a_1} \cdot \Delta a_1 + \frac{\partial f}{\partial a_2} \cdot \Delta a_2.$$

- So, when $\Delta a_i \in [\Delta_i, \Delta_i]$, the worst-case half-width in $a = f(a_1, a_2)$ is equal to

$$\Delta^t = \left|\frac{\partial f}{\partial a_1}\right| \cdot \Delta_1 + \left|\frac{\partial f}{\partial a_2}\right| \cdot \Delta_2.$$

- The result of applying dual interval arithmetic is

$$\Delta^d = \left|\left|\frac{\partial f}{\partial a_1}\right| \cdot \Delta_1 - \left|\frac{\partial f}{\partial a_2}\right| \cdot \Delta_2\right|.$$

- Thus, the average half-width – corresponding to random interval arithmetic – is equal to

$$\Delta^r = \max\left(\left|\frac{\partial f}{\partial a_1}\right| \cdot \Delta_1, \left|\frac{\partial f}{\partial a_2}\right| \cdot \Delta_2\right).$$

Title Page
◀◀ ▶▶
◀ ▶
Page 11 of 14
Go Back
Full Screen
Close
Quit

11. Computing $f(a_1, \ldots, a_n)$

- Similarly, for $n > 2$ variables, we conclude that

$$\Delta^r = \max\left(\left|\frac{\partial f}{\partial a_1}\right| \cdot \Delta_1, \ldots, \left|\frac{\partial f}{\partial a_n}\right| \cdot \Delta_n\right).$$

- In interval computations, we estimate the range of a function over a box $[\underline{a}_1, \overline{a}_1] \times \ldots \times [\underline{a}_n, \overline{a}_n]$.
- If a box is not too narrow, the estimates are too wide.
- To improve the estimates, we:
  - bisect the box along one of the directions and
  - repeat the estimation for each of the two half-boxes.
- The optimal direction in a direction a_i in which the product $\left|\frac{\partial f}{\partial a_i}\right| \cdot \Delta_i$ is the largest possible.
- The above value Δ^r is exactly the value of this maximum.

12. Relation with Fuzzy Logic

- *Our formula:*
$$\Delta_c = \max(\Delta_1, \Delta_b).$$

- *Fuzzy logic – objective:*
  - we know:
    * the degree of belief $a = d(A)$ in a statement A and
    * the degree of belief $b = d(B)$ in a statement B,
  - we want to estimate the degree of belief $c = d(C)$ in $C \stackrel{\text{def}}{=} A \vee B$.

- In the most widely used (and most practically successful version) of fuzzy logic,
$$c = \max(a, b).$$

13. Acknowledgments

This work was supported in part:

- by NASA under cooperative agreement NCC5-209,
- by the NSF grants EAR-0112968, EAR-0225670, and EIA-0321328, and
- by the NIH grant 3T34GM008048-20S1.

Title Page

◀◀ ▶▶

◀ ▶

Go Back

Full Screen

Close

Quit