# Using Known Relation Between Quantities to Make Measurements More Accurate and More Reliable


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## 1. One of the main objectives of science and engineering

- One of the main objectives of science and engineering is:
  - to predict the future state of the world, and
  - to make sure that this future state is as beneficial for us as possible.
- The state of the world at any given moment of time can be characterized by the corresponding values of physical quantities.
- Thus, what we need is to predict future values of these quantities.
- For example, predicting weather means predicting the future values of temperature, humidity, wind speed and direction, etc.

## 2. Measurements and resulting data processing are very important

- To make the desired predictions, we need to know the relation between:
  - each future value $y$ of the quantities of interest and
  - current values $x_1, \ldots, x_n$ of related quantities.
- This relation is often described in terms of differential (and other) equations.
- By applying an appropriate method to solve this equation, we get an algorithm $y = f(x_1, \ldots, x_n)$ for estimating $y$ from $x_i$.
- So, to come up with the desired estimate $\widetilde{y}$ for $y$, we:
  - measure the current values of the corresponding quantities $x_1 \ldots, x_n$, and then
  - we use the measurement results $\widetilde{x}_1, \ldots, \widetilde{x}_n$ to compute the estimate $\widetilde{y} = f(\widetilde{x}_1, \ldots, \widetilde{x}_n)$ for each desired future value $y$.

## 3. One of the main metrological questions: how accurate is the resulting estimate?

- Measurement results $\widetilde{x}_i$ are, in general, different from the actual (unknown) values $x_i$.
- There is a non-zero difference $\Delta x_i \stackrel{\text{def}}{=} \widetilde{x}_i - x_i$ known as *measurement error*.
- Because of these difference:
  - the estimate $\widetilde{y}$ is, in general, different from
  - the actual value $y = f(x_1, \ldots, x_n)$ – even when the function $f(x_1, \ldots, x_n)$ describes the relation between $x_i$ and $y$ exactly.
- The measurement uncertainty in $x_i$ *propagates* through the algorithm $f$.

## 4. One of the main metrological questions: how accurate is the resulting estimate (cont-d)

- A natural question is: how accurate is the estimate $\widetilde{y}$, i.e., what we can say about the difference $\Delta y \stackrel{\text{def}}{=} \widetilde{y} - y$.
- This is one of the main metrological questions, and in metrology, there are many efficient algorithms for propagating uncertainty.

## 5. Traditional metrology is very important

- In general, the more accurately we measure, the more accurate our predictions.
- In this process, traditional metrology is very valuable; it teaches us:
  - how to gauge the accuracy of the measuring instruments and even
  - how to calibrate the instrument so as to further increase its accuracy.

## 6. Need to take into account relation between quantities

- However, traditional metrology mostly concentrates on individual measurements.
- In practice, often:
  - in addition to the relation between future and present values,
  - there are also relations between the current values of different quantities,
  - and this relation is not always captured by the correlations used in traditional metrological analysis.
- For example, there is usually an known upper bound on the difference between the values of the same quantity:
  - at close moments of time or
  - at nearby locations.
- It is known that such relations can lead to more accurate measurement results.

## 7. Need to account for relation between quantities (cont-d)

- For example, suppose that we know that the values of the mechanical stress at two nearby locations are in intervals

$$[0.80, 1.00] \text{ and } [0.90, 1.10].$$

- Suppose that we know that the difference between the actual values cannot exceed 0.01.
- This means that the first quantity cannot be smaller than $0.90 - 0.01$, so we get a narrower interval $[0.89, 1.00]$.
- Similarly, we get a narrower interval $[0.90, 1.01]$ for the second quantity.
- In general, such relations take the form of inequalities or inequalities.
- It is well known that any inequality or equality constraint can be described by inequalities of the type $g_j(x_1, \ldots, x_n) \le 0$.

## 8. Need to account for relation between quantities (cont-d)

- A general inequality constraint $a(x_1, \ldots, x_n) \le b(x_1, \ldots, x_n)$ is equivalent to $a(x_1, \ldots, x_n) - b(x_1, \ldots, x_n) \le 0$.
- A general equality constraint $a(x_1, \ldots, x_n) = b(x_1, \ldots, x_n)$ is equivalent to two inequality constraints:

  $$a(x_1, \ldots, x_n) - b(x_1, \ldots, x_n) \le 0 \text{ and } b(x_1, \ldots, x_n) - a(x_1, \ldots, x_n) \le 0.$$

- For example, the above relation of the type $|x_i - x_j| \le \delta$ means that we have two inequalities: $x_i - x_j \le \delta$ and $x_i - x_j \ge -\delta$.
- They are equivalent to

  $$x_i - x_j - \delta \le 0 \text{ and } -\delta - (x_i - x_j) \le 0.$$

- Here, $g_1(x_1, \ldots, x_n) = x_i - x_j - \delta$ and $g_2(x_1, \ldots, x_n) = -\delta - (x_i - x_j)$.

## 9. Case of indirect relations

- So far, we have considered relations that directly relate the measured quantities.
- However, relations can also involve not-directly-measurable parameters of a model that describes the corresponding phenomenon.
- For example, we know that:
  - for a cantilever beam with a triangular line load linearly decreasing from $q_0$ at distance 0 to 0 at distance $z = \ell$,
  - the bending $x_i$ at location $z_i$ is determined by the following formula:

$$x_i = c_1 \cdot \left( -\frac{1}{120\ell} z_i^5 + \frac{1}{24} z_i^4 + \frac{1}{12} z_i^3 \cdot \ell + \frac{1}{12} z_i^2 \cdot \ell^2 \right).$$

- Here $c_1 \stackrel{\text{def}}{=} \dfrac{q_0}{E \cdot I}$.
- We measure the values $x_i$, but we do not directly the value $c_1$.

## 10. Case of indirect relations (cont-d)

- Since different measurement results are related to the same coefficient $c_1$, they are indirectly related to each other.
- Namely, we know the exact ratios $x_i/x_j$ of the bendings corresponding to different locations on the beam.
- In general, we can have models with several not-directly-measurable parameters $c_1, \ldots, c_k$.
- So a general constraint can take a form $g_j(x_1, \ldots, x_n, c_1, \ldots, c_k) \le 0$.
- Based on the measurements, we can find estimates $\widetilde{c}_i$ for these parameters.
- These estimates are based on measurements, and measurements are not absolutely accurate.
- So, these estimates, in general, differ from the actual (unknown) values of these parameters, i.e., we have a non-zero estimation errors

$$\Delta c_i \stackrel{\text{def}}{=} \widetilde{c}_i - c_i.$$

## 11. What we do in this talk

- In this talk, we analyze the general problem of using known relations between quantities.
- The purpose is to make measurements more accurate and more reliable.
- Our main focus is on the – rather typical practical case – when we only know the upper bound on the measurement error.
- Measurement errors are usually relatively small.
- So terms quadratic (or higher order) with respect to measurement errors can be safely ignored.
- We show that:
  - under this – usual – linearization assumption,
  - the corresponding problems can be effectively solved by known linear programming algorithms.

## 12. What we do in this talk (cont-d)

- We also consider the case when:
  - we know the probability distributions of different measurement errors,
  - we know the reliability of each measuring instrument, and
  - we know our degree of confidence in each relation between the quantities.
- In this case, the problem is computationally more difficult, but still solvable.

## 13. Usual approach to measurement uncertainty and its propagation: a brief reminder and why go beyond

- This talk focuses on situations in which there is a practical need to go beyond the usual approach to uncertainty propagation.
- To explain this need, we will first briefly remind the reader about the usual approach to measurement uncertainty and its propagation.
- This approach is described in the Guide to the Expression of Uncertainty in Measurement (GUM).

## 14. Usual approach to measurement uncertainty and its propagation (cont-d)

- To better explain why there are situations when we need to go beyond GUM:
  - we will not just reproduce the GUM formulas,
  - we will also remind the readers about the main assumptions behind these formulas.
- This will help us explain that there are situations:
  - when these assumptions are not satisfied and
  - where, moreover, the uncritical use of GUM formulas can lead to misleading results.

## 15. Main assumptions behind the usual GUM formulas

- The main GUM formulas – the ones that people mostly use – are based, in effect, on the following two assumptions:
- The measurement errors are relatively small – and thus, we can linearize the corresponding expressions
- We know the joint probability of all the measurement errors, i.e.:
  - we know the probability distributions of each measurement error (in particular, its mean and its standard deviation), and
  - we know the relation between these measurement errors.
- These assumptions are not always satisfied in practice.
- GUM also has recommendations on what to do in such cases. Let us recall these two assumptions.

## 16. Possibility of linearization

- Measurement errors are usually relatively small.
- So terms quadratic (or higher order) with respect to measurement errors are much smaller than the linear terms.
- For example:
  - even for a not very accurate measurement, for which the measurement error is about 10%,
  - the square of this error is 1%, which is much smaller than 10%.
- For more accurate measurements, the ratio of quadratic to linear terms is even smaller.
- So, we can:
  - safely ignore quadratic and higher order terms in the Taylor expansion of the corresponding dependencies, and
  - retain only linear terms.

## 17. Possibility of linearization (cont-d)

- Let us apply this idea to the error $\Delta y \stackrel{\text{def}}{=} \widetilde{y} - y$ in the result of processing data caused by measurement errors.
- Here, $\Delta y = \widetilde{y} - y = f(\widetilde{x}_1, \ldots, \widetilde{x}_n) - f(x_1, \ldots, x_n)$.
- By definition of the measurement error $\Delta x_i$, we get $x_i = \widetilde{x}_i - \Delta x_i$.
- Substituting this expression into the above formula, we conclude that

$$\Delta y = f(\widetilde{x}_1, \ldots, \widetilde{x}_n) - f(\widetilde{x}_1 - \Delta x_1, \ldots, \widetilde{x}_n - \Delta x_n).$$

- Expanding this expression in Taylor series in terms of $\Delta x_i$ and keeping only linear terms in this expansion, we conclude that

$$\Delta y = f_1 \cdot \Delta x_1 + \ldots + f_n \cdot \Delta x_n, \text{ where } f_i \stackrel{\text{def}}{=} \frac{\partial f}{\partial x_i}_{|x_1 = \widetilde{x}_1, \ldots, x_n = \widetilde{x}_n}.$$

## 18. How linearization helps to propagate uncertainty and what to do if measurement errors are not small: reminder

- we can find the probability distribution for $\Delta y$ by using:
  - the above formula and
  - the second assumption – that we know the joint probability distribution of the measurement errors.
- First, since we know the distribution, we thus know the mean value $E[\Delta x_i]$ (known as *bias*) of each measurement error.
- Thus, we can subtract this bias from all measurement results.
- For example, if we know that the clock runs, on average, 3 minutes ahead, we can subtract 3 from the clock's reading.
- This way, we get measurements whose bias is 0. In this case, due to the above formula, the bias of $\Delta y$ is also 0.

## 19. How linearization helps to propagate uncertainty and what to do if measurement errors are not small (cont-d)

- If we are only interested in the standard deviation $\sigma$ of $\Delta y$, then:
  - we can use the covariance matrix $c_{ij} \stackrel{\text{def}}{=} E[\Delta x_i \cdot \Delta x_j]$ – which we know since we know the distribution
  - to conclude that $\sigma^2 = \sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n} f_i \cdot f_j \cdot c_{ij}$.
- In particular, in frequent situations when measurement errors $\Delta x_i$ are independent, this formula takes a simplified form $\sigma^2 = \sum\limits_{i=1}^{n} f_i^2 \cdot \sigma_i^2$.
- Here $\sigma_i$ is the standard deviation of $\Delta x_i$.
- What if we are not satisfied with the standard deviation, and we want to know the full probability distribution for $\Delta y$?
- For this purpose, we can use Monte-Carlo simulation techniques.
- The same Monte-Carlo technique can be used if the measurement errors are not small, and thus, linearization is not possible.

## 20. Sometimes, we do not know the probability distributions of measurement errors

- One of the two main GUM assumptions is that the know the probability distribution of measurement errors.
- However, there are two types of situations in which we do not know these probabilities.
- The first is the most common type, when we are performing routine measurements – on the factory floor, in a building, etc.
- In principle, we can calibrate each sensor and get the probability distribution of its measurement error.
- However nowadays, most sensors are very cheap – unless we are looking for very accurate ones – while calibration is very expensive.
- As a result, most sensors used in industry and in practice are not calibrated very accurately.

## 21. Sometimes, we do not know the probability distributions of measurement errors (cont-d)

- For example, for a sensor that measures the body temperature, it is enough to know that it provides temperature with measurement error not exceeding $0.2^\circ$ C.
- It makes no sense to come up with the exact probability distribution.
- Another is the case of state-of-the-art measurements, when the usual calibration techniques are not applicable.
- Indeed, the usual sensor calibration techniques assume that:
  - there is a measuring instrument – called *standard*,
  - which is much more accurate than the sensor that we are trying to calibrate.
- However, if we perform the measurements with the most accurate sensor available, then this sensor is already the most accurate.
- No sensor is more accurate than the one we have.

## 22. Sometimes, we do not know the probability distributions of measurement errors (cont-d)

- In this case, the best we can do is to get some theory-justified upper bound on the measurement error.
- In both cases, we only know the upper bound $\Delta$ on the absolute value $|\Delta x|$ of the measurement error $\Delta x \stackrel{\text{def}}{=} \widetilde{x} - x$.
- The upper bound, by the way, is the absolute minimum information that we need to know;
  - if we do not even know the upper bound on the measurement error,
  - this means that the actual value $x$ can be as different from the measurement result $\widetilde{x}$ as possible;
  - this is not a measurement, this is a wild guess.

## 23. Sometimes, we do not know the probability distributions of measurement errors (cont-d)

- When all we know is the upper bound $\Delta$, then:
  - once we perform the measurement and get the measurement result $\widetilde{x}$,
  - the only information that we now have about the actual value $x$ of the measured quantity is that $x \in [\widetilde{x} - \Delta, \widetilde{x} + \Delta]$.
- Because of this, this type of uncertainty is known as *interval uncertainty*.

## 24. What is usually recommended in such cases

- In practice:
  - when all we know is the upper bound on the measurement error,
  - people often assume that the measurement error is uniformly distributed on the corresponding interval $[-\Delta, \Delta]$.
- This is what is usually recommended when we use the GUM formulas.
- This assumption makes sense:
  - if we have no reason to assume that some values from this interval are more probable than others,
  - then it is reasonable to assume that all the values are equally probable, i.e., that we have a uniform distribution.
- This argument goes back to Laplace and is known as *Laplace Indeterminacy Principle.*

## 25. What is usually recommended in such cases (cont-d)

- In more general cases, this principle leads to the Maximum Entropy approach, when:
  - out of all possible probability distributions with probability density $\rho(x)$,
  - we select the one for which the entropy $S \stackrel{\text{def}}{=} -\int \rho(x) \cdot \log(\rho(x))\, dx$ is the largest.
- In particular, this implies that:
  - if we have no information about the correlations between $\Delta x_i$,
  - it is reasonable to assume that these measurement errors are independent.

## 26. Why we need to go beyond this recommendation

- One of the main purposes of metrology is to produce *guaranteed* information about the measured values of different physical quantities.
- From this viewpoint, as we will show on a simple example, selecting a uniform distribution can lead to misleading conclusions.
- Let us consider the simplest possible relation between the desired quantity $y$ and the easier-to-measure quantities $x_1, \ldots, x_n$:

$$y = x_1 + \ldots + x_n.$$

- Let us also assume, for simplicity, that the measured values $\widetilde{x}_1, \ldots, \widetilde{x}_n$ of all these quantities are 0s.
- Let us also assume that for each of these measurements:
  - the only information that we have about the possible values of the measurement error $\Delta x_i = \widetilde{x}_i - x_i$
  - is that this value is bounded by a given positive number $\Delta$.

## 27. Why we need to go beyond this recommendation (cont-d)

- In this case, all we know about each actual value $x_i$ is that this value is located in the interval $[-\Delta, \Delta]$.
- In this situation, what can we say about possible values of $y$?
- Since each of the values $x_i$ is bounded from above by $\Delta$, the sum of $n$ such terms cannot exceed $n \cdot \Delta$.
- On the other hand, it is possible that each value $x_i$ is equal to $\Delta$, in which case their sum is exactly $n \cdot \Delta$.
- Similarly, since each of the values $x_i$ is bounded from below by $-\Delta$, the sum of $n$ such terms cannot be smaller than $-n \cdot \Delta$.
- On the other hand, it is possible that each value $x_i$ is equal to $-\Delta$, in which case their sum is exactly $-n \cdot \Delta$.
- So, the range of possible values of $y$ is the interval $[-n \cdot \Delta, n \cdot \Delta]$.
- Let us describe what the GUM formulas imply in this case.

## 28. Why we need to go beyond this recommendation (cont-d)

- Based on the above recommendation, we assume that:
  - measurement errors are independent, and
  - each measurement error is uniformly distributed on $[-\Delta, \Delta]$.
- Each of these distributions has mean 0 and variance $\sigma_i^2 = \Delta^2/3$.
- Here, $f_1 = \ldots = f_n = 1$, so the GUM formula leads to $\sigma^2 = n \cdot \Delta^2/3$.
- For large $n$, according to the Central Limit Theorem, the distribution of $y$ is close to Gaussian.
- For Gaussian distribution, with high confidence, we usually conclude that the actual value is in the interval $[m - k_0 \cdot \sigma, m + k_0 \cdot \sigma]$.
- For $k_0 = 3$, this is true with confidence 99.9%.
- For $k_0 = 6$, this is true with confidence $1 - 10^{-8}$.
- So, with high confidence, we conclude that the actual value $y$ is located in the interval $\left[-k_0 \cdot \sqrt{n} \cdot \frac{\Delta}{\sqrt{3}}, k_0 \cdot \sqrt{n} \cdot \frac{\Delta}{\sqrt{3}}\right]$.

## 29. Why we need to go beyond this recommendation (cont-d)

- So, the upper bound resulting from the uniform-distribution assumption is proportional to $\sqrt{n}$, while the actual bound grows as $n$.
- For large $n$, $\sqrt{n}$ is much smaller than $n$. So:
  - even in this very simple example,
  - the uniform-distribution assumption drastically underestimates the inaccuracy of the data processing result.

## 30. So what can we do in this case: interval computations

- Suppose that all we know about each value $x_i$ is that it is in the interval $[\widetilde{x}_i - \Delta_i, \widetilde{x}_i + \Delta_i]$.
- Then all we can say about $y = f(x_1, \ldots, x_n)$ is that $y$ belongs to the range

$$[\underline{y}, \overline{y}] \stackrel{\text{def}}{=} \{f(x_1, \ldots, x_n) : x_i \in [\widetilde{x}_i - \Delta_i, \widetilde{x}_i + \Delta_i] \text{ for all } i\}.$$

- Computing the endpoints $\underline{y}$ and $\overline{y}$ of this range is one of the main problems of *interval computations.*
- In particular, in the linearized case, we have $[\underline{y}, \overline{y}] = [\widetilde{y} - \Delta, \widetilde{y} + \Delta]$, where

$$\Delta = \sum_{i=1}^{n} |f_i| \cdot \Delta_i.$$

## 31. Remaining problem

- Usual interval computation techniques do not cover the situation when:
  - in addition to intervals $[\widetilde{x}_i - \Delta_i, \widetilde{x}_i + \Delta_i]$,
  - we also know some relation between the quantities $x_i$: e.g., that there is an upper bound $\delta$ on the difference between two values: $|x_i - x_j| \le \delta$.
- As we have shown in Section 1, in this case:
  - not only we get more accurate estimates for $y$,
  - we also get more accurate estimates for each of the measured quantities $x_i$.
- Let us analyze how this can be done in the general case.

## 32. How to use known relations between quantities to make measurements more accurate

- We measure $n$ quantities $x_1, \ldots, x_n$ and get $n$ measurement results

$$\widetilde{x}_1, \ldots, \widetilde{x}_n.$$

- For each measurement $i$, we know the upper bound $\Delta_i$ on the absolute values of the measurement error $\Delta x_i = \widetilde{x}_i - x_i$.
- In this case, we can conclude that each actual value $x_i$ is located in the interval $[\widetilde{x}_i - \Delta_i, \widetilde{x}_i + \Delta_i]$.
- Suppose that we also know the relations between these quantities

$$g_1(x_1, \ldots, x_n, c_1, \ldots, c_k) \le 0,$$

$$\ldots,$$

$$g_r(x_1, \ldots, x_n, c_1, \ldots, c_k) \le 0.$$

## 33. Possibility of linearization

- We have mentioned the possibility of linearizing $f(x_1, \ldots, x_n)$.
- Similarly, we can represent each constraint $g_j(x_1, \ldots, x_n, c_1, \ldots, c_k) \le 0$ as $g_j(\widetilde{x}_1 - \Delta x_1, \ldots, \widetilde{x}_n - \Delta x_n, \widetilde{c}_1 - \Delta c_1, \ldots, \widetilde{c}_k - \Delta c_k) \le 0$.
- Expanding this expression in Taylor series in terms of $\Delta x_i$ and keeping only linear terms in this expansion, we conclude that

$$\widetilde{g}_j + g_{j1} \cdot \Delta x_1 + \ldots + g_{jn} \cdot \Delta x_n +$$

$$m_{j1} \cdot \Delta c_1 + \ldots + m_{jk} \cdot \Delta c_k \le 0.$$

- Here we denoted $\widetilde{g}_j \stackrel{\text{def}}{=} g_j(\widetilde{x}_1, \ldots, \widetilde{x}_n, \widetilde{c}_1, \ldots, \widetilde{c}_k)$,

$$g_{ji} \stackrel{\text{def}}{=} -\frac{\partial g_j}{\partial x_i}_{|x_1=\widetilde{x}_1,\ldots,x_n=\widetilde{x}_n,c_1=\widetilde{c}_1,\ldots,c_k=\widetilde{c}_k}, \quad m_{ji} \stackrel{\text{def}}{=} -\frac{\partial g_j}{\partial c_i}_{|x_1=\widetilde{x}_1,\ldots,x_n=\widetilde{x}_n,c_1=\widetilde{c}_1,\ldots,c_k=\widetilde{c}_k}.$$

- Let us show how to use this info to get more accurate estimates for the quantities $x_1, \ldots, x_n$.

## 34. How we can solve this problem

- For each $i$, the smallest possible value of $x_i$ can be obtained by solving the following constrained optimization problem:
- Minimize $\widetilde{x}_i - \Delta x_i$ under the following constraints:

$$-\Delta_1 \le \Delta x_1 \le \Delta_1, \ldots, -\Delta_n \le \Delta x_n \le \Delta_n,$$

$$\widetilde{g}_1 + g_{11} \cdot \Delta x_1 + \ldots + g_{1n} \cdot \Delta x_n +$$

$$m_{11} \cdot \Delta c_1 + \ldots + m_{1k} \cdot \Delta c_k \le 0,$$

$$\ldots,$$

$$\widetilde{g}_1 + g_{r1} \cdot \Delta x_1 + \ldots + g_{jn} \cdot \Delta x_n +$$

$$m_{r1} \cdot \Delta c_1 + \ldots + m_{rk} \cdot \Delta c_k \le 0.$$

- To find the largest possible value of $x_i$, we need to maximize $\widetilde{x}_i - \Delta x_i$ under the same constraints.
- In both problems, we optimize the value of a linear expression under linear constraints.

## 35. How we can solve this problem (cont-d)

- Such problems are known as problems of *linear programming*.
- Several efficient algorithms are known for solving these problems.
- By using linear programming techniques, we can find:
  - the solution $\underline{x}_i$ to the minimization problem and
  - the solution $\overline{x}_i$ to the maximization problem.
- We can now conclude that the range of possible values of $x_i$ is equal to $[\underline{x}_i, \overline{x}_i]$.
- A simple example mentioned earlier shows that this can indeed lead to a drastic improvement of accuracy.

## 36. How to use known relations between quantities to make measurements more accurate: case of data processing

- We are interested in the value of a difficult-to-measure quantity $y$.
- This quantity is related to several easier-to-measure quantities $x_1, \ldots, x_n$ by a known relation $y = f(x_1, \ldots, x_n)$.
- We measure $n$ quantities $x_1, \ldots, x_n$ and get $n$ measurement results

$$\widetilde{x}_1, \ldots, \widetilde{x}_n.$$

- We plug in these measurement results into the algorithm $f$, and we get the estimate $\widetilde{y} = f(\widetilde{x}_1, \ldots, \widetilde{x}_n)$.
- What can we conclude about the possible values of the estimation error $\Delta y = \widetilde{y} - y$?
- We assume that for each measurement $i$, we know the upper bound $\Delta_i$ on the absolute values of the measurement error $\Delta x_i = \widetilde{x}_i - x_i$.
- In this case, the estimation error is determined by the above formula.

# 37. Case of data processing (cont-d)

- In the absence of any other information, all we can conclude about $\Delta y$ is that $|\Delta y| \le \Delta$, where

$$\Delta \stackrel{\text{def}}{=} |f_1| \cdot \Delta_1 + \ldots + |f_n| \le \Delta_n.$$

- Suppose now that we also know the relations between these quantities

$$g_1(x_1, \ldots, x_n, c_1, \ldots, c_k) \le 0,$$

$$\ldots,$$

$$g_r(x_1, \ldots, x_n, c_1, \ldots, c_k) \le 0.$$

- As we have mentioned earlier:
  - since the measurement errors are usually small,
  - we can describe these relations in the linearized form.
- How can we use this info to get more accurate estimates for $\Delta y$?

## 38. How we can solve this problem

- The smallest possible value of $\Delta y$ can be obtained by solving the following constrained optimization problem:
- Minimize $\widetilde{y} - (f_1 \cdot \Delta x_1 + \ldots + f_n \cdot \Delta x_n)$ under the above constraints.
- To find the largest possible value of $\Delta y$, we need to maximize the same expression under the same constraints.
- Both problems are particular cases of linear programming.
- So, we can apply efficient linear programming algorithms to solve these problems.
- We can use linear programming techniques and find:
  - the solution $\underline{y}$ to the minimization problem and
  - the solution $\overline{y}$ to the maximization problem.
- We can then conclude that the range of possible values of $y$ is equal to $[\underline{y}, \overline{y}]$.

## 39. How to use known relations between quantities to make measurements more reliable

- A sensor can malfunction.
- As a result, the value generated by this sensor will, in general, become very different from the actual value of the measured quantity.
- This leads to two natural questions.
- How can we detect that one of the sensors malfunctioned? If we can detect this, this will make the measurement results more reliable.
- Once we detect that one of the sensors malfunctioned, how can we determine which sensor malfunctioned?

## 40. How to detect that one of the sensors malfunctioned: analysis of the problem

- Suppose that, as in our first numerical example, that we measure two related quantities $x_1$ and $x_2$ by two sensors.
- For each of them, the upper bound on the absolute value of the measurement error is $\Delta_1 = \Delta_2 = 0.1$.
- Suppose that we know that the actual values $x_1$ and $x_2$ of two quantities cannot differ by more than 0.01: $|x_1 - x_2| \le 0.01$.
- Suppose that, as in our example, the first sensor produced the value $\widetilde{x}_1 = 0.9$.
- However, the second sensor malfunctioned and generated the value $\widetilde{x}_2 = 2.0$ – which is far away from the actual (unknown) value $x_2$.
- In this case, we cannot find two values $x_1$ and $x_2$ for which

$$|x_1 - 0.9| \le 0.1, \quad vert x_2 - 2.0| \le 0.1, \text{ and } |x_1 - x_2| \le 0.01.$$

## 41. How to detect that one of the sensors malfunctioned: analysis of the problem (cont-d)

- Indeed, in this case, $x_1 \leq 1.0$, $x_2 \geq 1.9$ and thus, the difference $x_2 - x_1$ is larger than or equal to $1.9 - 1.0 = 0.9$ – much larger than 0.01.
- This is a typical situation:
  - if one of the sensors malfunctions,
  - it is highly probable that the system of constraints can no longer be satisfied.
- So, we arrive at the following suggestion.

## 42. How to detect that one of the sensors malfunctioned: idea

- If the system of inequalities is inconsistent, this means that one of the sensors malfunctioned.
- Checking consistency of a system of linear inequalities can be done by the same efficient linear programming algorithms.

## 43. How can we determine which sensor malfunctioned

- If the system of inequalities (8) is no longer consistent, what we can do is try:
  - for each $i$ from 1 to $n$,
  - to delete all inequalities involving $\Delta x_i$ from the system and check whether the resulting reduced system is still consistent.
- When we delete a well-functioning sensor:
  - the system will still contain the inequalities coming from the malfunctioning sensor and
  - thus, the system will most probably still be inconsistent.
- However, when we delete the malfunctioning sensor, the remaining system will become consistent.
- So, we can determine the malfunctioning sensor as the one whose deletion makes the system consistent.

## 44. Comments

- On the qualitative level, this idea is far from being new: it is used a lot.
- For example:
  - if a thermometer show a temperature of 5° C in an office while in a neighboring office a thermometer shows 20° C,
  - this clearly means that one of these sensor malfunctioned.
- What if a car flashes a red light indicating that something may be wrong?
- Mechanics always compare with other measurement results to make sure that it is a real problem, and not a sensor malfunction.
- A similar idea can be used to detect whether the physical systems itself – whose quantities we are measuring – is malfunctioning.
- Usually, there are some thresholds for the corresponding quantities.

## 45. Comments (cont-d)

- For example, for a building, stresses should not exceed a certain level.
- For a chemical plant, temperature in the reactor must lie within given bounds, etc.
- If some values from the interval $[\underline{y}, \overline{y}]$ that we obtain get outside these bounds, this is an indication that something needs to be done.

## 46. Application to digital twins

- How can we understand how different effects and different controls will affect a system: be it an airplane, a building, a plant?
- A good idea is to design an accurate simulating program.
- Such program are known as *digital twins*.
- From the metrological viewpoint, there are two major challenges related to digital twins.
- The first challenge is related to the fact that we determine the parameters $c_1, \ldots, c_k$ of the corresponding model based on measurements.
- The value $c_i$ are determined based on measurements and are, thus, only known with some uncertainty.
- The estimated values $\widetilde{c}_i$ used in the design of the digital twin are, in general, somewhat different from the ideal best-fit values $c_i$.
- There is an estimation error $\Delta c_i = \widetilde{c}_i - c_i$.

## 47. Application to digital twins (cont-d)

- As a result, the values predicted by a digital twin are, in general, somewhat different from what we actually observe.
- However, the current digital twins do not provide us with any bounds on this difference.
- It is therefore desirable to make digital twins generate not the numerical values – as now – but rather intervals of possible values.
- The second major challenge is that the current digital twins do not learn.
- As we compare the predictions of the digital twin with how the actual system behaves, we get additional information.
- This information can potentially help us make the model more accurate.
- However, in the current digital twin technology, there is no easy way to do it.

## 48. Application to digital twins (cont-d)

- How can we make a digital twin that learns?
- There exist effective machine learning tools like deep learning.
- However, these tools require a lot of data to train, and we rarely have that much data about the actual physical system.
- So, what can we do?

## 49. How to make digital twins learn: main idea

- The formulas describing the digital twin can be formulated as constraints.
- For example, if the digital twin predict the value $x_i$ as $t_i(c_1, \ldots, c_k)$, for some algorithm $t_i$, this can be described as an equality

$$x_i = t_i(c_1, \ldots, c_k).$$

- Now, each measurement of the actual values of each physical quantity results in this equality constraint plus the usual constraint

$$|x_i - \widetilde{x}_i| \le \Delta_i.$$

# 50. How to make digital twins learn: main idea (cont-d)

- So:
  - to get a better constraint on the values of each of the parameters $c_i$,
  - we can maximize and minimize each value $c_i = \widetilde{c}_i - \Delta c_i$ under the corresponding constraints.
- Then:
  - as we add more and more measurement results,
  - the corresponding bounds $[\underline{c}_i, \overline{c}_i]$ will lead to more and more accurate description of the best-fit parameters $c_i$.

## 51. This will also help digital twins return intervals, and not just numbers without any estimates of their accuracy

- This way, digital twins will not only predict approximate estimates.
- We can also use the above-described ideas to generate:
  - the bounds on possible values of $t_i(c_1, \ldots, c_k)$
  - when each $c_i$ is in the corresponding interval.

## 52. How can we make this idea more practical

- In the above idea, each additional measurement adds new inequalities to the system of inequalities.
- As the number of inequalities grows, the computation time needed to solve this system of inequalities also grows.
- At some point, it may exceed the computational ability of the computational system.
- To make computations realistic, we can use the following natural idea:
- First, we perform a certain number $M$ of measurements.
- Then, we use the above scheme to find the better bounds $\underline{c}_i \le c_i = \widetilde{c}_i - \Delta c_i \le \overline{c}_i$ on the parameters $c_i$.
- Then, we again start measuring and comparing the measurement results with the predictions of the digital twin.

# 53. How can we make this idea more practical (cont-d)

- Once we have made $M$ new measurements, we form a system that takes into account:
  - not only these new $M$ measurements,
  - but also $k$ previously determined inequalities

$$\underline{c}_i \leq c_i = \widetilde{c}_i - \Delta c_i \leq \overline{c}_i.$$

- Solving this system of inequalities leads us to more accurate bounds $\underline{c}_i$ and $\overline{c}_i$.
- Then, we again perform $M$ new measurements, etc.
- In this case, the number of inequalities that we need to process remains limited.
- Alternatively, we can – if we have enough computational power:
  - repeat this procedure every time we get a new measurement,
  - but using only $M$ latest measurement results.

## 54. What if we have probabilistic uncertainty?

- In the previous sections, we consider the case when:
  - we only know the bounds on the measurement errors, and
  - we do not have any information about the probabilities of different values within these bounds.
- In many practical situations, however, we know the corresponding probability distributions.
- In such situations, how can we use known relations between the measured quantities to make measurements more accurate?
- One possibility is to use Monte-Carlo simulations; namely:
  - many times, we simulate probability distributions for all $n$ measurement errors $\Delta x_i$, but
  - then we dismiss all simulated tuples $(\Delta x_1, \ldots, \Delta x_n)$ that do not satisfy the inequalities describing the known relations.

## 55. What if we have probabilistic uncertainty (cont-d)

- Based on remaining tuples, we can get the histograms of the corresponding values $\Delta x_i$ and/or $\Delta y$.
- Thus, get a good understanding of the probability distributions that take into account the relations between the quantities $x_i$.
- What we propose is, in effect, a minor modification of the usual GUM-related practice:
  - instead of simply using the known joint probability distribution of the measurement errors,
  - we propose to use the related distribution on the tuples $x = (x_1, \ldots, x_n)$ of actual values.

## 56. What if we have probabilistic uncertainty (cont-d)

- This distribution takes into account:
  - not only the measurement errors,
  - but also the relation between the actual values of the quantities.
- In effect, this is similar to the GUM-recommended Bayesian approach.
- The difference is that we use the relation between the quantities instead of the prior distribution.

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