Decision Making Beyond Arrow's "Impossibility Theorem", with the Analysis of Effects of Collusion and Mutual Attraction

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1. Group Decision Making and Arrow's Impossibility Theorem

- In 1951, Kenneth J. Arrow published his famous result about group decision making.
- This result that became one of the main reasons for his 1972 Nobel Prize.
- *The problem:*
  - A group of n participants $P_1, \ldots, P_n$ needs to select between one of m alternatives $A_1, \ldots, A_m$.
  - To find individual preferences, we ask each participant P_i to rank the alternatives A_j from the most desirable to the least desirable:

$$A_{j_1} \succ_i A_{j_2} \succ_i \ldots \succ_i A_{j_n}.$$

  - Based on these n rankings, we must form a single group ranking (in the group ranking, equivalence $\sim$ is allowed).

2. Case of Two Alternatives Is Easy

- *Simplest case:*
  - we have only two alternatives A_1 and A_2,
  - each participant either prefers A_1 or prefers A_2.
- *Solution:* it is reasonable, for a group:
  - to prefer A_1 if the majority prefers A_1,
  - to prefer A_2 if the majority prefers A_2, and
  - to claim A_1 and A_2 to be of equal quality for the group (denoted $A_1 \sim A_2$) if there is a tie.

3. Case of Three or More Alternatives Is Not Easy

- *Arrow's result:* no group decision rule can satisfy the following natural conditions.
- *Pareto condition:* if all participants prefer A_j to A_k, then the group should also prefer A_j to A_k.
- *Independence from Irrelevant Alternatives:* the group ranking between A_j and A_k should depend only on how participants rank A_j and A_k – and should not depend on how they rank other alternatives.
- *Arrow's theorem:* every group decision rule which satisfies these two condition is a *dictatorship* rule:
  - the group accepts the preferences of one of the participants as the group decision and
  - ignores the preferences of all other participants.
- This clearly violates another reasonable condition of *symmetry*: that the group decision rules should not depend on the order in which we list the participants.

4. Beyond Arrow's Impossibility Theorem: Nash's Bargaining Solution

- *Usual claim:* Arrow's Impossibility Theorem is often cited as a proof that reasonable group decision making is impossible – e.g., that a perfect voting procedure is impossible.
- *Our claim:* Arrow's result is only valid if we have binary (partial) information about individual preferences.
- *Conclusion:* that the pessimistic interpretation of Arrow's result is, well, too pessimistic :-)
- *Implicit assumption behind Arrow's result:* the only information we have about individual preferences is the binary ("yes"-"no") preferences between the alternatives.
- *Fact:* this information does not fully describe a persons' preferences.
- *Example:* the preference $A_1 \succ A_2 \succ A_3$:
  - it may indicate that a person strongly prefers A_1 to A_2, and strongly prefers A_2 to A_3, and
  - it may also indicate that this person strongly prefers A_1 to A_2, and at the same time, A_2 is almost of the same quality as A_3.
- *How can this distinction be described:* to describe this degree of preference, researchers in decision making use the notion of *utility*.

5. Why Utility

- *Idea of value:* a person's rational decisions are based on the relative values to the person of different outcomes.
- *Monetary value is often used:* in financial applications, the value is usually measured in monetary units such as dollars.
- *Problem with monetary value:* the same monetary amount may have different values for different people:
  - a single dollar is likely to have more value to a poor person
  - than to a rich one.
- *Thus, a new scale is needed:* in view of this difference, in decision theory, to describe the relative values of different outcomes, researchers use a special *utility* scale instead of the more traditional monetary scales.

6. What Is Utility: a Reminder

- *Main idea behind utility:* a common approach is based on preferences of a decision maker among *lotteries*.
- *Specifics:*
  - take a very undesirable outcome A^- and a very desirable outcome A^+;
  - consider the lottery $A(p)$ in which we get A^+ with given probability p and A^- with probability $1-p$;
  - a utility $u(B)$ of an outcome B is defined as the probability p for which B is of the same quality as $A(p)$: $B \sim A(p) = A(u(B))$.
- *Assumptions behind this definition:*
  - clearly, the larger p, the more preferable $A(p)$:

$$p < p' \Rightarrow A(p) < A(p');$$

  - the comparison amongst lotteries is a linear order – i.e., a person can always select one of the two alternatives;
  - comparison is Archimedean – i.e. if for all $\varepsilon > 0$, an outcome B is better than $A(p-\varepsilon)$ and worse than $A(p+\varepsilon)$, then it is of the same quality as $A(p)$: $B \sim A(p)$.

7. Different Utility Scales

- *Comment:* as defined above, utility always takes values within the interval $[0, 1]$.
- *Possibility:* it is also possible to define utility to take values within other intervals.
- *Why this is possible:* indeed, the numerical value $u(B)$ of the utility depends on the choice of reference outcomes A^- and A^+.
- *Result:* the usual axioms of utility theory guarantee that two utility functions $u(B)$ and $u'(B)$ corresponding to different choices of the reference pair are related by a linear transformation:

$$u'(B) = a \cdot u(B) + b$$

for some real numbers $a > 0$ and b.

- *Conclusion:* by using appropriate values a and b, we can then re-scale utilities to make the scale more convenient.
- *Example:* in financial applications, we can make the scale closer to the monetary scale.

8. Expected Utility

- *Problem:*
  - *Situation:* we have n incompatible events $E_1, \ldots, E_n$ occurring with known probabilities $p_1, \ldots, p_n$; if E_i occurs, we get the outcome B_i.
  - *Examples of events:* coins can fall heads or tails, dice can show 1 to 6.
  - *We know:* the utility $u_i = u(B_i)$ of each outcome B_i.
  - *Find:* the utility of the corresponding lottery.
- *Solution:*
  - *Main idea:* $u(B_i) = u_i$ means that each B_i is equivalent to getting A^+ with probability u_i and A^- with probability $1 - u_i$.
  - *Conclusion:* the lottery "B_i if E_i" is equivalent to the following two-step lottery:
    * first, we select E_i with probability p_i, and
    * then, for each i, we select A^+ with probability u_i and A^- with the probability $1 - u_i$.
  - In this two-step lottery, the probability of getting A^+ is equal to

$$p_1 \cdot u_1 + \ldots + p_n \cdot u_n.$$

  - *Result:* the utility of the lottery "if E_i then B_i" is equal to

$$u = \sum_{i=1}^{n} p_i \cdot u_i = \sum_{i=1}^{n} p(E_i) \cdot u(B_i).$$

9. Nash's Bargaining Solution

- *How to describe preferences:* for each participant P_i, we can determine the utility $u_{ij} \stackrel{\text{def}}{=} u_i(A_j)$ of all the alternatives $A_1, \ldots, A_m$.
- *Question:* how to transform these utilities into a reasonable group decision rule?
- *Solution:* was provided by another future Nobelist John Nash.
- *Nash's assumptions:*
  - symmetry,
  - independence from irrelevant alternatives, and
  - *scale invariance*, i.e., invariance under replacing the original utility function $u_i(A)$ with an equivalent function $a \cdot u_i(A)$,
- *Nash's result:* the only group decision rule is selecting an alternative A for which the product $\prod\limits_{i=1}^{n} u_i(A)$ is the largest possible.
- *Comment.* the utility functions must be scaled in such a way that the "status quo" situation $A^{(0)}$ is assigned the utility 0, e.g., by replacing the original utility values $u_i(A)$ with re-scaled values $u'_i(A) \stackrel{\text{def}}{=} u_i(A) - u_i(A^{(0)})$.

10. Properties of Nash's Solution

- *Nash's solution satisfies the Pareto condition:*
  - If all participants prefer A_j to A_k, this means that $u_i(A_j) > u_j(A_k)$ for every i,
  - hence $\prod\limits_{i=1}^{n} u_i(A_j) > \prod\limits_{i=1}^{n} u_i(A_k)$, which means that the group would prefer A_j to A_k.
- *Nash's solution satisfies the Independence condition:*
  - According to Nash's solution, we prefer A_j to A_k if $\prod\limits_{i=1}^{n} u_i(A_j) > \prod\limits_{i=1}^{n} u_i(A_k)$.
  - From this formula, once can easily see that the group ranking between A_j and A_k depends only on how participants rank A_j and A_k.
- *Nash's solution can be easily explained in terms of fuzzy logic:*
  - We want all participants to be happy.
  - So, we want the first participant to be happy *and* the second participant to be happy, etc.
  - We can take $u_1(A)$ as the "degree of happiness" of the first participant, $u_2(A)$ as the "degree of happiness" of the second participant, etc.
  - In order to formalize "and", we use the operation $d \cdot d'$ (one of the two operations originally proposed by L. Zadeh to describe "and").
  - Then, the degree to which all n participants are satisfied is equal to the product $u_1(A) \cdot u_2(A) \cdot \ldots \cdot u_n(A)$.

11. How We Can Determine Utilities

- *General idea:* use the iterative *bisection* method in which, at every step, we have an interval $[\underline{u}, \overline{u}]$ that is guaranteed to contain the actual (unknown) value of the utility u.

- *Starting interval:* in the standard scale, $u \in [0, 1]$, so we can start with the interval $[\underline{u}, \overline{u}] = [0, 1]$.

- *Iteration:* once we have an interval $[\underline{u}, \overline{u}]$ that contains u, we:
  - compute its midpoint $u_{\text{mid}} \stackrel{\text{def}}{=} (\underline{u} + \overline{u})/2$, and
  - compare the alternative A_j with the lottery

$$L \stackrel{\text{def}}{=} \text{“}A^+ \text{ with probability } u_{\text{mid}}, \text{ otherwise } A^-\text{”}.$$

- Depending on the result of this comparison, we can now halve the interval $[\underline{u}, \overline{u}]$:
  - If, for the participant, the alternative A_j is better than this lottery $A_j \succ L$, then we know that $u \in [u_{\text{mid}}, \overline{u}]$, so we have a new interval $[u_{\text{mid}}, \overline{u}]$ of half-width which is guaranteed to contain u.
  - If $A_j \prec L$, then we know that $u \in [\underline{u}, u_{\text{mid}}]$, so we have a new interval $[\underline{u}, u_{\text{mid}}]$ of half-width which is guaranteed to contain u.

- After each iteration, we decrease the width of the interval $[\underline{u}, \overline{u}]$ by half.

- Thus, after k iterations, we get an interval of width 2^{-k} which contains the actual value u – i.e., we have determined u with the accuracy 2^{-k}.

12. Problem: Sometimes It Is Beneficial to Cheat

- *Assumption:* the above description relies on the fact that we can elicit true preferences (and hence, true utility functions) from the participants.
- *Problem:* sometimes, it is beneficial for a participant to cheat and provide false utility values.
- *Example:*
  - Suppose that the participant P_1 know the utilities of all the other participants.
  - An ideal situation for P_1 is when, out of m alternatives $A_1, \ldots, A_m$, the group as a whole selects an alternative A_{m_1} which is the best for P_1, i.e, for which $u_1(A_{m_1}) \geq u_1(A_j)$ for all $j \neq m_1$.
  - It is not necessarily true that the product $\prod\limits_{i=1}^{n} u_i(A_j)$ computed based on P_1's true utility is the largest for the alternative A_{m_1}.
  - However, we can force this product to attain the maximum for A_{m_1} if we report, e.g., a "fake" utility function $u_1'(A)$ for which $u_1'(A_{m_1}) = 1$ and $u_1'(A_j) = 0$ for all $j \neq m_1$.

13. In Case of Uncertainty, Cheating May Hurt the Cheater: an Observation

- *In the above example:*
  - one person is familiar with the preferences of all the others,
  - while others have no information about this person's preferences.
- *Usual case:* if other participants have no information about this person's preferences, then this person has no information about the preferences of the others as well.
- *Our claim:* in this case, cheating may be disadvantageous.
- *Example:*
  - Let P_1 use the above false utility function $u'_1(A)$ for which $u'_1(A_{m_1}) = 1$ and $u'_1(A_j) = 0$ for all $j \neq m_1$.
  - *Possibility:* others have similar utility functions with $u_i(A_{m_i}) > 0$ for some $m_i \neq m_1$ and $u_i(A_j) = 0$ for all other j.
  - Then for every alternative A_j, Nash's product is equal to 0.
  - From this viewpoint, all alternatives are equally good, so each of them can be chosen.
  - In particular, it may be possible that the group selects an alternative A_q which is *the worst* for P_1 – i.e., for which $u_1(A_q) < u_1(A_j)$ for all $j \neq p$.

14. Case Study: Territorial Division

- We have a set A to divide.
- Each alternative corresponds to a partition of the set A into n subsets $X_1, \ldots, X_n$ such that $\bigcup\limits_{i=1}^{n} X_i = A$ and $X_i \cap X_j = \emptyset$ when $i \neq j$.
- The utility functions have the form $u_i(X) = \int_X v_i(t)\,dt$ for given functions $v_i(t)$ from the set A to the set of non-negative real numbers.
- Based on the utility functions $v_i(t)$, we find a partition $X_1, \ldots, X_n$ for which Nash's product $u_1(X) \cdot \ldots \cdot u_n(X_n)$ attains the largest possible value.
- *Choices:* the participant P_1 can either report his/her actual function $v_1(t)$, or he/she can report a different utility function $v_1'(t) \neq v_1(t)$.
- *Assumption:* P_1 does not know others' utility functions.
- For each reported function $v_1'(t)$, we can find the partition $X_1, \ldots, X_n$ that maximizes the corresponding product

$$\left(\int_{X_1} v_1'(t)\,dt\right) \cdot \left(\int_{X_2} v_2(t)\,dt\right) \cdot \ldots \cdot \left(\int_{X_n} v_n(t)\,dt\right).$$

- *Question:* which utility function $v_1'(t)$ should the participant P_1 report in order to maximize his gain $u(v_1', v_1, v_2, \ldots, v_n) = \int_{X_1} v_1'(t)\,dt$?

15. Decision Making under Uncertainty: a Reminder

- When deciding on v_1, the participant P_1 must make a decision under uncertainty.
- The situation of decision making under uncertainty is typical in decision making.
- We can choose an *optimistic* approach in which, for each action A, we only consider its most optimistic outcome, with the largest possible gain $u^+(A)$ – and choose an action for which this luckiest outcome is the largest.
- Alternatively, we can choose a *pessimistic* approach in which, for each action A, we only consider its most pessimistic outcome, with the smallest possible gain $u^-(A)$ – and choose an action for which this worst-case outcome is the largest.
- Realistically, both approaches appear to be too extreme.
- In real life, it is more reasonable to use, as an objective function, Hurwicz's combined pessimism-optimism criterion:
  - we choose a real number $\alpha \in [0,1]$, and
  - choose an alternative A for which the combination

$$u(A) = \alpha \cdot u^-(A) + (1-\alpha) \cdot u^+(A)$$

takes the largest possible value.

16. For Territorial Division, It Is Beneficial to Report the Correct Utilities: Result

- For Hurwicz's $u(A) = \alpha \cdot u^-(A) + (1-\alpha) \cdot u^+(A)$,
  - pessimism $u(A) = u^+(A)$ corresponds to $\alpha = 1$,
  - optimism $u(A) = u^+(A)$ corresponds to $\alpha = 0$,
  - realistic approaches correspond to $\alpha \in (0, 1)$.
- *Comment:* while Hurwicz's combination may sound arbitrary, it is actually the only rule which satisfied reasonable scale-invariance conditions.
- For our problem, Hurwicz's criterion means that we select a utility function $v_1'(t)$ for which the combination

$$J(v_1') \stackrel{\text{def}}{=} \alpha \cdot \min_{v_2,\ldots,v_n} u(v_1', v_1, v_2, \ldots, v_n) + (1-\alpha) \cdot \max_{v_2,\ldots,v_n} u(v_1', v_1, v_2, \ldots, v_n)$$

  attains the largest possible value.
- *Result:* when $\alpha > 0$, the objective function $J(v_1')$ attains its largest possible value for $v_1'(t) = v_1(t)$.
- *Conclusion:* unless we select the optimistic criterion, it is always best to select $v_1'(t) = v_1(t)$, i.e., to tell the truth.

17. How to Find Individual Preferences from Collective Decision Making: Inverse Problem of Game Theory

- *Problem:* in some cases, however, we have a subgroup ("clique") of participants who do their best to make joint decisions and who do not want to disclose their differences.
- *Example:* this is a frequent situation, e.g., within political groups – who are afraid that any internal differences can be exploited by the competing groups.
- *Challenge:* in such situations, it is extremely difficult to determine individual preferences based on the group decisions.
- *Example:* during the Cold War, this is what kremlinologists tried to do – with different degrees of success.
- *Assumptions:*
  - We have a group of n participants $P_1, \ldots, P_n$ that does not want to reveal its individual preferences.
  - We can ask the group as a whole to compare different preferences.
  - We assume that when making group decisions, the group uses Nash's solution.

18. Comments

- *Fact:* Nash's solution depends only on the product of the utility functions.
- *Corollary:* in the best case,
  - we will be able to determine n individual utility functions
  - without knowing which of these functions corresponds to which individual.
- *Comment:* this is OK, because
  - the main objective of our determining these utility functions is to be able to make decision of a larger group based on Nash's solution,
  - and in making this decision, it is irrelevant who has what utility function.
- *Comparing with political analysts:* in this sense, our problem is easier than the problem solved by political analysts:
  - from our viewpoint, it is sufficient to know that one member of the ruling clique is more conservative and another is more liberal, but
  - a political analyst would also be interesting in knowing who exactly is conservative and who is more liberal.

19. How to Find Individual Preferences from Collective Decision Making: Our Main Result

- Let n be the number of participants and m be the number of alternatives.
- By a *lottery*, we mean a vector $p = (p^{(0)}, p^+, p_1, \ldots, p_m)$ for which $p_j \ge 0$ and $p^{(0)} + p^+ + p_1 + \ldots + p_m = 1$.
- *Comment.* Here, the probability $p^{(0)}$ means the probability of the status quo state $A^{(0)}$, p^+ means the probability of the outcome A^+, and the utilities are scaled in such a way that for each participant, $u_i(A^{(0)}) = 0$ and $u_i(A^+) = 1$.
- By an *individual utility function*, we mean a vector $(u_1, \ldots, u_m)$, $u_j \ge 0$.
- By a *group utility function*, we mean a collection of n utility functions

$$(u_{i1}, u_{i2}, \ldots, u_{im}).$$

- We say that a group utility function u *leads* to the following preference relation $<$ between the lotteries: $p < q$ if and only if

$$\prod_{i=1}^{n} \left(p^+ + \sum_{j=1}^{m} p_j \cdot u_{ij} \right) < \prod_{i=1}^{n} \left(p^+ + \sum_{j=1}^{m} q_j \cdot u_{ij} \right).$$

- *Our result.* If two group utility functions u_{ij} and u'_{ij} lead to the same preference, then they differ only by permutation.
- So, we can determine individual preferences from group decisions.
- An efficient algorithm for this determination is given in the paper.

20. We Must Take Altruism and Love into Account

- *Implicit assumption:* so far, we (implicitly) assumed that the utility $u_i(A_j)$ of a participant P_i depends only on what he or she gains.
- *In real life:* the degree of a person's happiness also depends on the degree of happiness of other people.
  - Normally, this dependence is positive, i.e., we feel happier if other people are happy.
  - However, negative emotions such as jealousy are also common, when someone else's happiness makes a person not happy.
- The idea that a utility of a person depends on utilities of others was developed by another future Nobelist Gary Becker.
- *General description:* the utility u_i of i-th person is equal to $u_i = f_i(u_i^{(0)}, u_j)$, where $u_i^{(0)}$ is the utility that does not take interdependence into account, and u_j are utilities of other people.
- *Linear approximation: idea.* The effects of interdependence can be illustrated on the example of linear approximation, when we approximate the dependence by the first (linear) terms in its expansion into Taylor series.
- *Linear approximation: formulas.* The utility u_i of i-th person is equal to

$$u_i = u_i^{(0)} + \sum_{j \neq i} \alpha_{ij} \cdot u_j,$$

where the interdependence is described by the coefficients a_{ij}.

21. Paradox of Love

- *Reminder:* The utility u_i of i-th person is equal to $u_i = u_i^{(0)} + \sum\limits_{j \neq i} \alpha_{ij} \cdot u_j$, where $u_i^{(0)}$ is the utility that does not take interdependence into account.
- *Case of 2 persons:*

$$u_1 = u_1^{(0)} + \alpha_{12} \cdot u_2; \quad u_2 = u_2^{(0)} + \alpha_{21} \cdot u_1.$$

- *Example:* mutual affection means that $\alpha_{12} > 0$ and $\alpha_{21} > 0$.
- *Example:* selfless love, when someone else's happiness means more than one's own, corresponds to $\alpha_{12} > 1$.
- *general solution:*

$$u_1 = \frac{u_1^{(0)} + \alpha_{12} \cdot u_2^{(0)}}{1 - \alpha_{12} \cdot \alpha_{21}}; \quad u_2 = \frac{u_2^{(0)} + \alpha_{21} \cdot u_1^{(0)}}{1 - \alpha_{12} \cdot \alpha_{21}}.$$

- *Paradox:* when two people are deeply in love with each other ($\alpha_{12} > 1$ and $\alpha_{21} > 1$), then positive original pleasures $u_i^{(0)} > 0$ lead to $u_i < 0$ – i.e., to unhappiness.
- *Comments.*
  - This phenomenon may be one of the reasons why people in love often experience deep negative emotions.
  - From this viewpoint, a situation when one person loves deeply and another rather allows him- or herself to be loved may lead to more happiness than mutual passionate love.

22. Why Two and not Three?

- An *ideal love* is when each person treats other's emotions almost the same way as one's own, i.e., $\alpha_{12} = \alpha_{21} = \alpha = 1 - \varepsilon$ for a small $\varepsilon > 0$.
- For *two people*, from $u_i^{(0)} > 0$, we get $u_i > 0$ – i.e., we can still have happiness.
- Suppose now that we have *three* (or more) people in the state of mutual affection, i.e., if
$$u_i = u_i^{(0)} + \alpha \cdot \sum_{j \neq i} u_j.$$
- *Simplifying assumption:* everyone gains the same $u_i^{(0)} = u^{(0)} > 0$.
- *Conclusion:*
$$u_i = u^{(0)} + (1 - \varepsilon) \cdot (n - 1) \cdot u_i,$$
hence
$$u_i = \frac{u^{(0)}}{1 - (1 - \varepsilon) \cdot (n - 1)} < 0,$$
i.e., we have unhappiness.
- This may be the reason why 2-person families are the main form.
- In other words, if two people care about the same person (e.g., his mother and his wife), all there of them are happier if there is some negative feeling (e.g., jealousy) between them.

23. Emotional vs. Objective Interdependence

- It is important to distinguish between:
  - *emotional interdependence* in which one's utility is determined by the utility of other people, and
  - *"objective" altruism*, in which
    - one's utility depends on the material gain of other people –
    - but not on their subjective utility values,

  i.e., in which (in the linearized case)

$$u_i = u_i^{(0)} + \sum_j \alpha_j \cdot u_j^{(0)}.$$

- In the objective approach we care about:
  - others' well-being and
  - not their emotions.
- In this approach, no paradoxes arise: any degree of altruism only improves the situation.
- The objective approach to interdependence was proposed and actively used by yet another Nobel Prize winner Amartya K. Sen.

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