### Computing Degrees of Subsethood and Similarity for Interval-Valued Fuzzy Sets: Fast Algorithms

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## 1. Subsethood and Set Equality – Important Notions of Set Theory

- In traditional set theory, among the basic notions are the notions of set equality and subsethood:
  - two sets A and B are equal if they contain exactly the same elements, and
  - a set A is a *subset* of the set B if every element of the set A also belongs to B.
- It is desirable to generalize these notions to fuzzy sets.
- In traditional set theory, for every two sets A and B, either  $A \subseteq B$  or  $A \not\subseteq B$ ; either A = B or  $A \neq B$ .
- The main idea behind fuzzy logic is that for fuzzy, imprecise concepts, everything is a matter of degree.
- Thus, it is reasonable to define degrees of subsethood  $d_{\subseteq}(A, B)$  and similarity  $d_{=}(A, B)$ .

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### How to Describe Degree of Subsethood

- In fuzzy logic and fuzzy set theory, there is no built-in degree of subsethood or degree of equality.
- In fuzzy, we only have union  $\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$ and intersection  $\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x)).$
- So, let us express subsethood in terms of  $\cup$  and  $\cap$ .
- Such an expression is well known in set theory: (1) in general,  $A \cap B \subseteq A$ , and (2)  $A \subseteq B \Leftrightarrow A \cap B = A$ .
- So, we can take the ratio  $d_{\subseteq}(A, B) = \frac{|A \cap B|}{|A|}$ , where |A| is the cardinality of A.
- For a fuzzy set,  $|A| = \sum_{x} \mu_A(x) = \sum_{i=1}^{n} a_i$ , so  $d_{\subseteq}(A,B) = \frac{\sum \min(a_i, b_i^x)}{\sum a_i}.$

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#### 3. Degree of Equality (Similarity)

- It is known that
  - in general,  $A \cap B \subseteq A \cup B$ , and
  - $-A = B \Leftrightarrow A \cap B = A \cup B.$
- So, it is reasonable to define the degree of similarity as  $d_{=}(A,B) = \frac{|A \cap B|}{|A \cup B|}.$
- The smaller the ratio, the more there are elements from one of the sets which do not belong to the other.
- A similar definition can be used to define degree of equality (similarity) of two fuzzy sets A and B:

$$d_{=}(A, B) = \frac{\sum_{i=1}^{n} \min(a_i, b_i)}{\sum_{i=1}^{n} \max(a_i, b_i)}.$$

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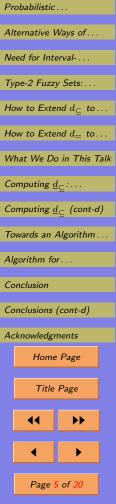




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# 4. Towards the Probabilistic Justification of the Above Formulas: Fuzzy Sets as Random Sets

- *Idea*: we can gauge  $\mu_A(x)$  as the proportion of experts who believe that x satisfies the property A.
- So,  $\mu_A(x)$  is the probability that, according to a randomly selected expert, x satisfies A.
- Every expert has a set of values that, according to this expert's belief, satisfy the property A.
- We consider the experts to be equally valuable, so these sets are equally probable.
- Thus, we have, in effect, a probability distribution on the class of all possible sets a random set.
- Thus,  $\mu_A(x)$  can be interpreted as the probability that a given element x belongs to the random set.



### Probabilistic Interpretation of the Formula for the Degree of Subsethood

- $A \subseteq B$  if and only if  $P(B \mid A) = 1$ .
- Thus, we define  $d_{\subseteq}(A,B) \stackrel{\text{def}}{=} P(B \mid A) = \frac{P(A \cap B)}{P(A)}$ .
- We know the probability  $\mu_A(x) = P_A(x \in S)$  that a given element x belongs to the random set.
- Thus,  $P(A) = \sum_{x} p(x) \cdot P_A(x \in S) = \sum_{x} p(x) \cdot \mu_A(x)$ .
- We assume: all x are equally probable: p(x) = c.
- Similarly,  $P(A \cap B) = c \cdot \sum \mu_{A \cap B}(x)$ , so

$$P(B \mid A) = \frac{c \cdot \sum_{x} \mu_{A \cap B}(x)}{c \cdot \sum_{x} \mu_{A}(x)} = \frac{\sum_{x} \mu_{A \cap B}(x)}{\sum_{x} \mu_{A}(x)} = \frac{\sum_{i=1}^{n} \min(a_{i}, b_{i})}{\sum_{i=1}^{n} a_{i}}.$$

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- $\bullet$  A = B if and only if that every element of the union
- So, the sets A and B are equal if

$$P(A \cap B \mid A \cup B) = \frac{P(A \cap B)}{P(A \cup B)} = 1.$$

 $A \cup B$  also belongs to the intersection  $A \cap B$ .

- Thus, we can define  $d_{=}(A, B) \stackrel{\text{def}}{=} P(A \cap B \mid A \cup B)$ .
- For fuzzy sets, we get  $P(A \cap B) = c \cdot \sum \mu_{A \cap B}(x)$  and  $P(A \cup B) = c \cdot \sum_{x} \mu_{A \cup B}(x).$
- Thus, the desired ratio takes the form

$$P(A \cap B \mid A \cup B) = \frac{c \cdot \sum_{x} \mu_{A \cap B}(x)}{c \cdot \sum_{x} \mu_{A \cup B}(x)} = \frac{\sum_{i=1}^{n} \min(a_i, b_i)}{\sum_{i=1}^{n} \max(a_i, b_i)}.$$

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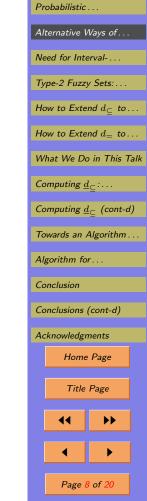
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### 7. Alternative Ways of Describing Degrees of Subsethood and Similarity: Comment

- The above expressions are the simplest and probably most frequently used.
- However, there exist other expressions.
- For example, we can use the following idea: (1) in general,  $B \subseteq A \cup B$ , and (2)  $B = A \cup B \Leftrightarrow A \subseteq B$ .
- Thus, as an alternative degree of subsethood, we can take a ratio

$$\frac{|B|}{|A \cup B|} = \frac{\sum_{i=1}^{n} b_i}{\sum_{i=1}^{n} \max(a_i, b_i)}.$$

• In this talk, we only consider the main definitions.



## 8. Need for Interval-Valued (and More General Type-2) Fuzzy Sets

- An expert often cannot describe his or her knowledge by an exact value or by a precise set of possible values.
- Instead, the expert describe this knowledge by using words from natural language.
- In fuzzy logic, the expert's degree of certainty that "x is A" is a number from  $\mu_A(x) \in [0, 1]$ .
- However, an expert is unable to meaningfully express his or her degree of certainty by a precise number.
- It is much more reasonable to assume that the expert will describe these degrees also by words.
- Thus, for every x, the degree  $\mu_A(x)$  is not a number, but rather a new fuzzy set.

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### 9. Type-2 Fuzzy Sets: Successes and Challenges

- Fact: type-2 fuzzy sets provide a more adequate representation of expert knowledge.
- Corollary: they lead to higher quality control, clustering, etc.
- Main challenge: transition to type-2 fuzzy sets leads to an increase in computation time:
  - in type-1, each degree  $\mu_A(x)$  is a number;
  - in type-2, to describe a fuzzy set  $\mu_A(x)$ , we need more than one number.
- Simplest case: each degree is an interval  $[\underline{\mu}_A(x), \overline{\mu}_A(x)]$ .
- Complexity: we only need 2 numbers to describe each degree.
- Corollary: interval-valued fuzzy sets are mostly widely among type-2 sets.

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### 10. How to Extend $d_{\subseteq}$ to Interval-Valued Fuzzy Sets: Formulation of the Problem

- For each i from 1 to n, we know the intervals  $[\underline{a}_i, \overline{a}_i]$  and  $[\underline{b}_i, \overline{b}_i]$  of possible membership degrees.
- For each combination of values  $a_i \in [\underline{a}_i, \overline{a}_i]$  and  $b_i \in [\underline{b}_i, \overline{b}_i]$ , we can compute the subsethood degree

$$d\subseteq = \frac{\sum_{i=1}^{n} \min(a_i, b_i)}{\sum_{i=1}^{n} a_i}.$$

- The objective is to find the range  $[\underline{d}_{\subseteq}, \overline{d}_{\subseteq}]$  of possible values of the above subsethood degree, i.e.,
  - to compute the smallest possible value  $\underline{d}_{\subseteq}$  of  $d_{\subseteq}$  when  $a_i \in [a_i, \overline{a}_i]$  and  $b_i \in [b_i, \overline{b}_i]$ ; and
  - to compute the largest possible value  $\overline{d}_{\subseteq}$  of  $d_{\subseteq}$  when  $a_i \in [a_i, \overline{a}_i]$  and  $b_i \in [b_i, \overline{b}_i]$ .

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## 1. How to Extend $d_{=}$ to Interval-Valued Fuzzy Sets: Precise Formulation of the Problem

- For each i from 1 to n, we know the intervals  $[\underline{a}_i, \overline{a}_i]$  and  $[\underline{b}_i, \overline{b}_i]$  of possible membership degrees.
- For each combination of values  $a_i \in [\underline{a}_i, \overline{a}_i]$  and  $b_i \in [\underline{b}_i, \overline{b}_i]$ , we can compute the similarity degree

$$d_{=} = \frac{\sum_{i=1}^{n} \min(a_{i}, b_{i})}{\sum_{i=1}^{n} \max(a_{i}, b_{i})}.$$

- The objective is to find the range  $[\underline{d}_{-}, \overline{d}_{-}]$  of possible values of the above similarity degree, i.e.,
  - to compute the smallest possible value  $\underline{d}_{-}$  of  $d_{-}$  when  $a_i \in [a_i, \overline{a}_i]$  and  $b_i \in [b_i, \overline{b}_i]$ ; and
  - to compute the largest possible value  $\overline{d}_{=}$  of  $d_{=}$  when  $a_i \in [a_i, \overline{a}_i]$  and  $b_i \in [b_i, \overline{b}_i]$ .

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#### 12. What We Do in This Talk

- In the paper: we design four fast algorithms:
  - a fast algorithm for computing  $\underline{d}_{\subset}$ ;
  - a fast algorithm for computing  $\overline{d}_{\subseteq}$ ;
  - a fast algorithm for computing  $\underline{d}_{=}$ ;
  - a fast algorithm for computing  $\overline{d}_{=}$ .
- How fast: each of these algorithms requires  $O(n \cdot \log(n))$  computation steps.
- In the talk: we describe, in detail, one of these algorithms: for computing  $\underline{d}_{C}$ .

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### Computing $\underline{d}_{\subseteq}$ : Analysis of the Problem

- Fact:  $d_{\subseteq} = \frac{\sum \min(a_j, b_j)}{\sum a_i}$  is increasing in  $b_i$ .
- Conclusion: minimum is attained when  $b_i = \underline{b}_i$ .
- Two possible cases:
  - (1)  $a_i \leq b_i$  hence  $\min(a_i, \underline{b}_i) = a_i$ ;
  - (2)  $\underline{b}_i \leq a_i$  hence  $\min(a_i, \underline{b}_i) = \underline{b}_i$ .
- Case 1:  $d \subseteq \frac{a_i + m_i}{a_i + M_i}$ , where  $m_i \stackrel{\text{def}}{=} \sum_{i \neq j} \min(a_j, \underline{b}_j)$  and  $M_i \stackrel{\text{def}}{=} \sum_{i \neq i} a_j.$
- Since  $\min(a_j, \underline{b}_i) \leq a_j$  for all j, we have  $m_i \leq M_i$ .
- Thus,  $d_{\subseteq} = 1 \frac{M_i m_i}{a_i + M_i}$  increases when  $a_i$  increases.
- Hence, the minimum is when  $a_i$  is the smallest:  $a_i = \underline{a}_i$ .

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### 14. Computing $\underline{d}_{\subset}$ (cont-d)

• Case 2 (
$$\underline{b}_i \le a_i$$
):  $d \subseteq \frac{m}{M} = \frac{\sum \min(a_j, b_j)}{\sum a_j} = \frac{m}{a_i + M_i}$ .

- Conclusion:  $d \subseteq \min \text{ when } a_i = \max = \overline{a}_i$ .
- Case 1:  $a_i = \underline{a}_i$  and  $\underline{a}_i \leq \underline{b}_i$ ,
- Case 2:  $a_i = \overline{a}_i$  and  $\underline{b}_i \leq \overline{a}_i$ .
- If  $\overline{a}_i < \underline{b}_i$ , Case 2 is impossible, so  $a_i = \underline{a}_i$ .
- If  $\underline{b}_i < \underline{a}_i$ , Case 1 is impossible, so  $a_i = \overline{a}_i$ .
- In the remaining cases, when  $\underline{a}_i \leq \underline{b}_i \leq \overline{a}_i$ , both  $a_i = \underline{a}_i$  and  $a_i = \overline{a}_i$  are possible.
- The value  $a_i = \underline{a}_i$  minimizes if replacing it with  $\overline{a}_i$  increase the ratio:  $\frac{m}{M} \leq \frac{m + (\underline{b}_i \underline{a}_i)}{M + (\overline{a}_i \underline{a}_i)}$ .
- So,  $a_i = \underline{a}_i \Leftrightarrow \frac{m}{M} \le \frac{m + (\underline{b}_i \underline{a}_i)}{M + (\overline{a}_i \underline{a}_i)} \Leftrightarrow \frac{m}{M} \le \frac{\underline{b}_i \underline{a}_i}{\overline{a}_i \underline{a}_i}$ .

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### 15. Towards an Algorithm for Computing $\underline{d}_{\subset}$

- If we know the optimal value  $\underline{d}_{\subseteq}$  of the ratio m/M, then, as we have shown:
  - for each i,
  - we can determine whether the minimum is attained for  $a_i = \underline{a}_i$  or for  $a_i = \overline{a}_i$ .
- It is sufficient to know where m/M is w.r.t n ratios  $r_i \stackrel{\text{def}}{=} \frac{\underline{b}_i \underline{a}_i}{\overline{a}_i \underline{a}_i}$ , i.e.,  $r_{(j)} \leq \frac{m}{M} \leq r_{(j+1)}$  for  $r_{(1)} \leq r_{(2)} \leq \dots$
- For each possible location  $r_{(j)} \leq m/M \leq r_{(j+1)}$ :
  - we compute the corresponding values  $a_i$ , and then
  - we compute the resulting ratio  $r^{(j)}$ .
- $\bullet$  Out of the resulting n+1 ratios, we find the smallest.
- This smallest ratio is returned as  $\underline{d}_{\subseteq}$ .

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$$I^{-} = \{i : \overline{a}_i < \underline{b}_i\}, I^{+} = \{i : \underline{b}_i < \underline{a}_i\},$$
  
$$I = \{1, \dots, n\} - I^{-} - I^{+}.$$

- For all  $i \in I$ , we compute the ratios  $r_i \stackrel{\text{def}}{=} \frac{\underline{b}_i \underline{a}_i}{\overline{a}_i \underline{a}_i}$  and sort them:  $r_1 \leq r_2 \leq \ldots \leq r_k$ .
- We then compute

$$m^{(0)} = \sum_{i \in I} \underline{a}_i + \sum_{i \in I^-} \underline{a}_i + \sum_{i \in I^+} \underline{b}_i; \quad M^{(0)} = \sum_{i \in I} \underline{a}_i + \sum_{i \in I^-} \underline{a}_i + \sum_{i \in I^+} \underline{b}_i;$$
$$m^{(j+1)} = m^{(j)} + (\underline{b}_{i+1} - \underline{a}_{i+1}); \quad M^{(j+1)} = m^{(j)} + (\overline{a}_{i+1} - \underline{a}_{i+1}).$$

- For all j from 0 to k+1, we compute  $r^{(j)} = \frac{m^{(j)}}{M^{(j)}}$ .
- The smallest of  $r^{(j)}$  is the desired smallest value  $\underline{d}_{\subset}$ .
- Sorting takes  $O(n \cdot \log(n))$  steps, rest is linear time O(n), so overall we need  $O(n \cdot \log(n))$  steps.

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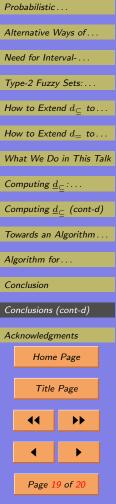
- To adequately capture commonsense reasoning, we must capture its imprecise character.
- One of the most successful ways to describe such reasoning is the technique of fuzzy sets.
- In fuzzy sets, for each element a, there is a degree  $\mu_A(a) \in [0, 1]$  to which a belongs to the set A.
- These degrees, in turn, can only be determined with uncertainty.
- So, in practice, we only know intervals  $[\underline{\mu}_A(a), \overline{\mu}_A(a)]$  of possible values of these degrees.
- In other words, we practice, we only have an intervalvalued fuzzy set.
- Among the most important concepts of set theory are the notions of subsethood and equality.

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#### 18. Conclusions (cont-d)

- Reminder: among the most important concepts of set theory are the notions of subsethood and equality.
- Thus,
  - to extend set theoretic techniques to fuzzy sets and interval-valued fuzzy sets,
  - we must be able to efficiently compute degrees of subsethood and degrees of equality (similarity):
    - \* for fuzzy sets and
    - $\ast$  for interval-valued fuzzy sets.
- There exist efficient algorithms for computing these degrees for fuzzy sets.
- We have shown how to extend these algorithms to a (more realistic) case of interval-valued fuzzy sets.



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