From Type-2 Fuzzy to Type-2 Intervals and Type-2 Probabilities

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1. Uncertainty Is Ubiquitous

- What are the main objectives of science and engineering?
  - We want to know the current state of the world.
  - We want to predict the future state of the world.
  - We want to find out how to make the future state of the world better.
- To describe the state of the world, we need to describe the values of the corresponding physical quantities.
- For example, in celestial mechanics, we need to know the positions and velocities of all celestial bodies.
- Similarly, to describe appropriate actions, we need to describes the values of the parameters of these actions.
- For example, we need to know the initial velocity and orientation of the spaceship that will lead us to Mars.

2. Uncertainty Is Ubiquitous (cont-d)

- Our information about the values of physical quantities come from measurements and from expert estimates.
- Measurements are never 100% accurate, there is always some uncertainty.
- For example, Vladik's height is 169.5 cm.
- This does not means that it is exactly 169.5000, it means ± 0.5.
- Expert estimates are usually even less certain.
- So, we always have uncertainty.

3. How Can We Describe This Uncertainty

- Uncertainty means that instead of a *single* value, we have the whole *set* of possible values of a quantity.
- This set is usually connected, i.e., it is an *interval*.
- In some cases, we also know the frequency with which, in similar situations, we encounter different values from this interval.
- This is known as *probabilistic* uncertainty.
- In many practical cases, we do not know the probabilities.
- However, an expert can estimate the degree to which different values are possible.
- This corresponds to *fuzzy* uncertainty.

4. Need to Take Uncertainty into Account When Processing Data

- We do not just measure quantities, we perform some computations $y = f(x_1, \ldots, x_n)$ with the measurement results $x_1, \ldots, x_n$.
- Due to uncertainty, the actual values x_i are, in general, different from the measurement results $\widetilde{x}_i$.
- Because of this:
  - the result $\widetilde{y} = f(\widetilde{x}_1, \ldots, \widetilde{x}_n)$ of processing measurement results is, in general, different from
  - the value $y = f(x_1, \ldots, x_n)$ that we would have gotten if we knew the exact values x_i.
- It is important to gauge the resulting uncertainty in y.

5. Need to Take Uncertainty into Account When Processing Data (cont-d)

- For example, if $\widetilde{y} = 100$ million tons is the estimated amount of oil in a given region, then:
  - if it is 100 ± 20, we should start exploiting, but
  - it it is 100 ± 200, maybe there is no oil at all, so we better perform additional measurement before wasting resources.
- For interval, probabilistic, and fuzzy uncertainty there are techniques for propagating uncertainty through computations.

6. Need for Type-2 Fuzzy

- As we have mentioned, an expert is cannot describe his/her estimate of the quantity of interest by a single number.
- Instead, the expert can produce an interval – or an interval with degrees assigned (= a fuzzy set).
- Similarly, the same expert cannot describe his/her degree of confidence $m(x)$ that x is possible by a single number.
- For example:
  - an expert can distinguish between degrees of confidence 0.7 and 0.8,
  - but hardly anyone can distinguish between degrees 0.80 and 0.81.
- It is more reasonable to expect that $m(x)$ is represented by an interval, or even by a fuzzy set.

7. Need for Type-2 Fuzzy (cont-d)

- So, for each x from the original interval $[\underline{x}, \overline{x}]$:
  - instead of a numerical degree of confidence $m(x)$,
  - we have an interval $[\underline{m}(x), \overline{m}(x)]$ – or a fuzzy set.
- This situation is known as *type-2 fuzzy*.
- Type-2 fuzzy sets – and especially interval-valued fuzzy sets – are well-studied and used in many applications.

8. Need for Type-2 Intervals and Probabilities, both Subjective and Objective

- Intervals and probabilities also come from observations and measurements.
- Observations and measurements always have uncertainty.
- Thus, we know intervals and probabilities with uncertainty too.
- Similarly to type-2 fuzzy, we can call methods that take this uncertainty into account *type-2*.
- There is an important difference between type-2 fuzzy and type-2 probabilities:
  - fuzzy (and type-2 fuzzy) information is *subjective*, while
  - interval and probabilistic uncertainty can be both *subjective* and *objective*.

9. Subjective and Objective Intervals and Type-2 Intervals: What Is Known

- In some cases, there exist an exact value, but we only know the interval containing this value.
- In this case, we have a subjective interval.
- In other cases, we have a range of values.
- E.g., person's height and weight changes during the day.
- So, the actual answer to the question "What is your height" is an interval.
- Since measurements are not absolutely accurate, we do not know the exact endpoints ℓ and r of this interval.
- Instead, we have:
  - an interval $[\underline{\ell}, \overline{\ell}]$ of possible values of ℓ, and
  - an interval $[\underline{r}, \overline{r}]$ of possible values of r.

10. Subjective and Objective Intervals and Type-2 Intervals (cont-d)

- This type-2 situation can be described by two nested intervals:
  - values from the inner interval $[\overline{\ell}, \underline{r}]$ *are* actual, while
  - values from the outer interval $[\underline{\ell}, \overline{r}]$ *may be* actual, but we are not sure.
- If we know that $y = f(x_1, \ldots, x_n)$ and we have such type-2 interval about each x_i, what can we say about y?
- If all uncertainties are independent, i.e., if all combinations of possible values of x_i are possible, then:
  - to find the inner interval for y, we apply interval computations to inner intervals for x_i, and
  - to find the outer interval for y, we apply interval computations to outer intervals for x_i.

11. Subjective Type-2 Probabilities: What Is Known

- Subjective type-2 probabilities mean that we only have partial information about the corresponding probabilities.
- This is known as *imprecise probabilities.*
- In general, as we mentioned, the basic type of uncertainty is interval uncertainty.
- In line with this, the basic type of probabilistic uncertainty is interval-valued probabilities.
- One of the main ways to describe a probability distribution is by a cumulative distribution function (cdf) $F(x) \stackrel{\text{def}}{=} \text{Prob}(X \leq x)$.
- A natural idea is thus to consider, for each x, an interval $[\underline{F}(x), \overline{F}(x)]$ of possible values of $F(x)$.
- This is known as *probability box*, or *p-box*, for short.
- p-boxes have been successfully used in many applications – as well as fuzzy-valued probabilities.

12. Objective Type-2 Probabilities: What Are They

- How can we have objective uncertainty in probability values?
- To understand this, let us recall what is probability from the practical viewpoint.
- In practice, probability p means, in effect, a frequency.
- We have a large number N of similar events (e.g., flipping a coin).
- These can be similar events occurring at different location and/or at different times.
- Probability p of a certain outcome means that this outcome is observed in $\approx p \cdot N$ cases.
- An ideal case is when the event settings are absolutely identical.

13. Objective Type-2 Probabilities (cont-d)

- For example:
  - we have a large set of identical atoms of a radioactive element, and
  - we observe how many of them emit radiation during a given period of time.
- In the usual quantum description, all the atoms are identical.
- However, the true quantum description is more complex.
- In this sense, quantum physics is similar to fuzzy:
  - one of the main ideas about fuzzy is Zadeh's statement that "everything is a matter of degree";
  - in quantum physics, the main idea is that everything is a matter of probability.

14. Objective Type-2 Probabilities (cont-d)

- In the first approximation – traditional quantum mechanics:
  - particle locations and velocities are only known with probabilities,
  - they can fluctuate around their classical values,
  - but the forces between particles are described by the usual formulas, e.g., Coulomb law

$$F = -c \cdot \frac{q_1 \cdot q_2}{r^2}.$$

- In secondary quantization, we take into account that the forces can also fluctuate around the classical values.
- In other words, the fields – that describe these forces – are also quantum objects whose values are only known with some probabilities.
- In general, no matter what kind of events we consider, these events are not identical.
- There are always quantum fluctuations because of which, for each event, the probability p_i is slightly different from p.

15. Objective Type-2 Probabilities (cont-d)

- Here, the values p_i are randomly fluctuating around the classical value p.
- In other words, here, we have objective type-2 probabilities.
- What does this mean in terms of observations?
- Can we experimentally detect the difference between type-1 and type-2 probabilities?
- To answer this question, let us recall what randomness means in terms of observations.

16. What Does Randomness Mean in Terms of Observations: Reminder

- What does randomness mean in terms of observations?
- Randomness means more than frequency.
- For example, according to Central Limit Theorem, differences between frequency and probability should be normally distributed.
- The general idea is that if a sequence is random, it must satisfy all the probability laws.
- A probability law is something that happens with probability 1.
- In mathematical terms, it is a set of probability measure 1 – so that its complement has measure 0.

17. What Does Randomness Mean in Terms of Observations (cont-d)

- So, a sequence is random if:
  - it does not belong to any definable set of probability measure 0,
  - or, equivalently, it does not belong to the union of all definable sets of measure 0.
- This is Kolmogorov's definition of a random sequence.
- Every definable set is described by a finite text – its definition.
- There are only countably many texts, so there are only countably many definable sets.
- The union of countably many sets of measure 0 still has measure 0.
- So, almost all sequence are random.

18. So Can We Experimentally Detect the Difference Between Type-1 and Type-2 Probabilities?

- We are interested in a sequence of events.
- Let $n_i = 1$ is the selected outcome occurred and $n_i = 0$ if it did not.
- We compare two cases:
  - type-1 case when each n_i occurs with probability p, and
  - type-2 case when each n_i occurs with probability p_i.
- Here:
  - we select some distribution on the set of all probabilities with mean p, and
  - take, as p_i, a random sequence of independent values corresponding to these probabilities.

19. Can We Experimentally Detect the Difference Between Type-1 and Type-2 Probabilities (cont-d)

- For both sequences, we can compare moments, i.e., averages over i from 1 to N of products $n_i^{a_1} \cdot n_{i+i_1}^{a_2} \cdot \ldots$.
- For example, mean is the average of n_i, correlation with next neighbor is average of $n_i \cdot n_{i+1}$, etc.
- Our *first result* is that for both sequences, each moment tends to the same limit: e.g., the mean tends to p.
- However, our *second result* is that for no sequence can be random with respect to both type-1 and type-2 distributions.
- This means that there are probability laws that are only true for type-1 sequences.
- So, it *is* possible to experimentally detect the difference!

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