When Is a Safe Freeway Lane Change Possible? Relativity Principle Explains an Empirical Formula

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1. Practical problem

- Lane change is the most safety-critical part of freeway driving – whether we deal with human driving or with self-driving cars.
- Before we start a lane change, it is important to make sure that:
  - this change is, in the current driving situation, indeed possible,
  - i.e., whether a gap between the cars of the next lane is sufficient for a safe lane change maneuver.
- Several techniques have been proposed for this determination.
- At present, the most effective is a fuzzy-based system proposed in 2025.
- This system improves upon the fuzzy-based system proposed in 2018.
- All current systems use the following empirical formula:
- The smallest size f of the safe gap exponentially depends on the freeway speed v:
$$d = a \cdot \exp(b \cdot v).$$

2. Practical problem (cont-d)

- The coefficients a and b, in general, depend on the specific freeway and specific road conditions.
- This formula is applicable in usual situations:
  - when all the cars on all the lanes move at the same speed,
  - the maximum speed allowed on this road segment.
- There are also versions of this formula that can be used:
  - for more complex situations,
  - when some of the lanes are congested and thus slower than others.
- These version also use exponential dependence.

3. Remaining problem

- Because of the safety-critical aspect of this practical problem, it is desirable to make sure that:
  - all the foundations for the used algorithms are well-justified,
  - and not just come from a purely computational extrapolation of a few empirical data.
- One of the main foundations for the existing lane-change techniques is the above exponential formula.
- It is therefore desirable:
  - either to provide, if possible, a theoretical foundation for this formula,
  - or, if such a foundation is not possible, to provide a alternative formula that does have a theoretical foundation.

4. What we do

- In this talk, we provide a theoretical foundation for this empirical formula.
- Somewhat unexpectedly, this foundation is based on:
  - the relativity principle
  - something that is, at first glance, as far away from our practical problem as possible.

5. Relativity principle: a brief reminder

- To most people, the relativity principle is associated with Einstein's Relativity Theory.
- This theory is important for describing objects moving with the speeds close to the speed of light.
- At first glance:
  - it has nothing to do with a car on a freeway
  - since the car's speed – around 100 km/h – is much much smaller than the speed of light, which is 300,000 km/sec.
- However, the relativity principle is not limited to close-to-speed-of-light motions.
- This principle was known many centuries before Einstein.

6. Relativity principle (cont-d)

- It has been first formulated by Galileo who noticed that:
  - for a passenger in a windowless ship cabin, when the sea is smooth,
  - it is not possible to tell whether the ship is moving or not – all the physical processes remain the same.
- This principle is perfectly valid for most processes described by Newton's mechanics.
- In Newton's mechanics, the only process that violated this principle is light.
- For all other processes, when we replace the stationary observer with a moving one, the observed speeds change.
- For example, when we are on a platform at a train station, we can see that the train is moving.
- However, when we are inside the train, we do not feel this motion.

7. Relativity principle (cont-d)

- In contrast to that, the famous Michelson-Morley experiments showed that the speed of light remains the same:
  - whether we measure it from the viewpoint of a stationary observer
  - or whether we measure it from the viewpoint of a moving observer.
- This clearly showed that for speeds close to the speed of light, Newtonian physics is not applicable.
- To come up with a reasonable adjustment of Newton's physics, Einstein decided to keep as many fundamental properties as possible.
- In particular, he decided to keep relativity ptinciple.
- The resulting theory indeed turned out to be a great success.
- Of course, in our practical situation, we are interested:
  - not in close-to-speed-of-light effects of relativity principle,
  - but in its low-speed applications.

8. How is relativity principle relevant for freeway driving

- The relation between the freeway driving and relativity principle is straightforward.
- When you drive on a street, you realize that you are driving, because your position with respect to the houses changes.
- However, suppose that you drive on a reasonably straight segment of a multi-lane freeway when it is:
  - separated by a high barrier from the surrounding city – to decrease noise – and
  - separated from the cars driving in the opposite direction – to avoid collisions.
- Then, the picture that you see remains the same, nothing changes.
- All the cars travel with the same speed – the maximum speed allowed on this road segment.

9. How is relativity principle relevant for freeway driving (cont-d)

- So, when you glance to the left and to the right, all the cars in your lane and on the neighboring lanes seem to freeze in the same position.
- Nothing seems to move.
- Of course, your engine hums, and the speedometer shows that you are going at high speed.
- However, without that, you would have had a full illusion that no one actually moves.
- This is not just a hypothetical illusion.
- In the movies, scenes that are supposed to be in a moving car are often shot when the car is actually standing still.
- In the resulting movie, one gets a full impression of a moving car.

10. How is relativity principle relevant for freeway driving (cont-d)

- So, are all the cars staying still or are all of them driving with the same speed v?
- Without irrelevant factors like noise or speedometer, we would not have noticed the difference, the physics is the same.

11. Let us use this relation to analyze how d depends on v

- For our practical problem, the relativity principle means, in particular, that: the desired relation $d = f(v)$ should not change if:
  - instead of the stationary coordinate system,
  - we use a coordinate system that moves with some velocity v_0.
- When we change to such a system, all velocity values change from v to $v' = v + v_0$.
- In these terms, invariance means that if we have $d = f(v)$ then for $v' = v + v_0$ we should have $d' = f(v')$.
- Notice that here we used d' and not the original value d.
- Indeed, what remains the same is physics, and not necessarily the numerical values of the corresponding physical quantities.

12. Let us use this relation to analyze how d depends on v (cont-d)

- For example, this is the whole idea of a wind tunnel – in which smaller-size models of airplanes were tested:
  - many aerodynamic processes are the same at different scales,
  - but on small scale, not only the sizes change, we also need to corresponding re-scale time intervals, wind speeds, etc.
- So, in general, instead of the original formula $d = f(v)$, we should have a formula $d' = f(v')$.
- Here, d' may be a re-scaled length, i.e., length described in different units.
- Suppose that we change the unit for measuring length from the original one to the one that is c times smaller.
- Then instead of the original numerical value d we get the new numerical values $d' = c \cdot d$.

13. Let us use this relation to analyze how d depends on v (cont-d)

- For example, when we replace meters with centimeters, then 1.7 m becomes $1.7 \cdot 100 = 170$ cm.
- So, we arrive at the following particular case of the relativity principle:

For each v_0, for all v and d, if $d = f(v)$, then $d' = f(v')$, where $v' = v + v_0$ and $d' = c \cdot d$, for some c depending on v_0.

14. This property explains the exponential dependence

- Substituting $d' = c(v_0) \cdot d$ and $v' = v + v_0$ into the formula $d' = f(v')$, we conclude that $f(v + v_0) = c(v_0) \cdot d$.
- Since $d = f(v)$, we thus conclude that $f(v + v_0) = c(v_0) \cdot f(v)$.
- In principle, there are some weird discontinuous functions $f(v)$ that satisfy this equation for some $c(v_0)$.
- However, it makes sense to assume that small changes in the velocity v should lead to small changes in the safe gap size d.
- Un mathematical terms, this means that the function $f(v)$ is continuous.
- In this case, all the solutions to the above functional equation are known.
- They are exactly the exponential dependencies.
- So, we have indeed arrived at a theoretical explanation for the empirical formula.

15. Comment

- The derivation of the exponential formula is easy if we also assume that the dependence $f(v)$ is smooth (differentiable).
- Indeed, in this case, the function $c(v_0) = f(v + v_0)/f(v)$ is also differentiable, as the ratio of two differentiable functions.
- Thus, we can differentiate both sides of the formula $f(v + v_0) = c(v_0) \cdot f(v)$ with respect to v_0, and then take $c_0 = 0$.
- This way, we get $f'(v) = c_0 \cdot f(v)$, where $f'(v)$ means derivative and $c_0 \stackrel{\text{def}}{=} c'(0)$.
- So, $df/dv = c_0 \cdot f$.
- If we divide both sides by f and multiply both sides by dv, we get an equivalent equality in which variables are separated: $df/f = c_0 \cdot dv$.
- Integrating both sides, we get $\ln(f) = c_0 \cdot v + c_1$, where c_1 is the integration constant.

16. Comment (cont-d)

- Reminder: we get $\ln(f) = c_0 \cdot v + c_1$, where c_1 is the integration constant.
- Applying $\exp(z)$ to both sides, we get the desired expression

$$f(v) = \exp(c_0 \cdot v + c_1) = \exp(a_0 \cdot v) \cdot \exp(c_1) = a \cdot \exp(b \cdot v).$$

- Here, $a \stackrel{\text{def}}{=} \exp(c_1)$ and $b \stackrel{\text{def}}{=} c_0$.

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