Why Best-Worst Method Works Well

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1. Formulation of the practical problem

- In many application areas, we rely on human estimates of different quantities.
- For example, when police investigates a crime, they rely on witnesses' estimates of the suspect's height and/or weight.
- In general:
  - we have n objects, and
  - for each object i, $i = 1, \ldots, n$, we want to know the corresponding value a_i of a quantity a.
- Estimates $\widetilde{a}_i$ of untrained people are usually not very accurate – and thus, not very helpful.
- What we humans are much better at is *comparing* different values.

2. Formulation of the practical problem (cont-d)

- For example:
  - if we see two people, especially if we see them side by side,
  - then we can conclude that one of them is, e.g., 20% taller than the other.
- Similarly:
  - an instructor may not be able to accurately predict how exactly each student will perform on a test, but
  - usually, instructors can predict who will do better and who will do worse, and how better and how worse.
- So, for some pairs (i, j), we ask the user to estimate the ratio $\dfrac{a_i}{a_j}$ of the corresponding values.
- Based on these estimates, we want to reconstruct the values of the desired quantities.

3. Comment

- To get more accurate estimates, we can do two things.
- We can ask the same person several times to compare the same pairs of objects.
- Then, for each pair of objects, we get the average of the resulting estimate.
- We can also ask several persons and get an average of their estimates.
- This way of getting a more accurate estimate is known as the paradigm of crowd wisdom.

4. Practical limitation

- In general, the more information we have, the more accurate the resulting estimate.
- From this viewpoint, the more questions we ask about different pairs, the better.
- However, as the number of objects increases, the number of pairs increases quadratically, as $\frac{n \cdot (n-1)}{2} \sim n^2$.
- For large n, it becomes un-realistic to ask questions about all the pairs.
- With such possibility in mind, it is necessary to ask the smallest possible number of questions.
- A natural idea is:
  - to select one of the objects i_0, and
  - to only ask for ratios between this object and all other objects.

5. Empirical fact and what we do in this talk

- It has been empirically shown that to get the most accurate estimates, we need:
  - either to compare all the quantities with the smallest one,
  - or to compare all the quantities with the largest one.
- This is known as the *best-worst method*.
- In this talk, we provide a theoretical explanation for this empirical result.

6. What we mean by reconstructing the values a_i

- In order to formulate the problem in precise terms, let us first clarify what we mean by reconstructing the values a_i.
- Of course, if we only know the ratios, we cannot uniquely determine the actual values.
- Indeed:
  - if we multiply all the values a_i by the same constant c,
  - then the ratios remain the same, while
  - the numerical values change.
- To avoid this non-uniqueness, a natural idea is to select some object i_0 for which we simply take $a_{i_0}^{\text{new}} \stackrel{\text{def}}{=} \widetilde{a}_{i_0}$.
- This means, in effect, that we replace the original measuring unit with a new one, which is $\dfrac{\widetilde{a}_{i_0}}{a_{i_0}}$ times smaller than the original measuring unit.

7. What we mean by reconstructing the values a_i (cont-d)

- In terms of this unit, the new values a_i^{new} of the desired quantity take the form

$$a_i^{\text{new}} = a_i \cdot \frac{\widetilde{a}_{i_0}}{a_{i_0}}.$$

- Since we multiply all the values of the quantity by the same constant $c = \dfrac{a_{i_0}^{\text{new}}}{a_{i_0}}$, the ratios remain the same:

$$\frac{a_i^{\text{new}}}{a_j^{\text{new}}} = \frac{a_i}{a_j}.$$

- A natural question is: which object i_0 should we select?

8. Once we selected i_0, how can we reconstruct the values a_i?

- If our estimates of the ratios were exact, then, in principle, by comparing all the objects with the selected object i_0, we could get the exact values of all the quantities a_i:
  - either as $a_i = \dfrac{a_i}{a_{i_0}} \cdot a_{i_0}$
  - or alternatively, as $a_i = \left(\dfrac{a_{i_0}}{a_i}\right)^{-1} \cdot a_{i_0}$.
- In practice, we do not know the exact ratios $\dfrac{a_i}{a_j}$, we only know the estimates w_{ij} for these ratios: $w_{ij} \approx \dfrac{a_i}{a_j}$.
- So, by using these estimates instead of the actual ratios, we can provide estimates $\widetilde{a}_i$ for the desired quantity by using:
  - either the formula $\widetilde{a}_i = \widetilde{a}_{i_0} \cdot w_{ii_0}$,
  - or, alternatively, the formula $\widetilde{a}_i = \widetilde{a}_{i_0} \cdot w_{i_0 i}^{-1}$.

9. But is there a difference between these two approaches?

- At first glance, it may look like it does not matter what method we use.
- Indeed, the estimated ratios $\dfrac{a_i}{a_{i_0}}$ and $\dfrac{a_{i_0}}{a_i}$ are simply inverses to each other.
- So a consistent person should select estimates which are inverses as well, i.e., estimates for which

$$w_{ii_0} = \frac{1}{w_{i_0 i}}.$$

- However, it is well known that people are not perfectly consistent.
- So, in general, these two estimates will lead to results:
  - which are not exactly mutually reverse and
  - which, thus, may lead to different estimates for the values a_i of the desired quantity.

10. Need to take uncertainty into account

- In practice, as we have mentioned, we can only estimate the ratios with some accuracy.
- Let us denote the accuracy with which we estimate the ratios by ε.
- This can be the mean squared value of the difference between the actual ratio $\dfrac{a_i}{a_j}$ and our estimate w_{ij}.
- This corresponds, e.g., to the probabilistic approach to uncertainty.
- This can also be the largest possible absolute value of this difference

$$\left|\frac{a_i}{a_j} - w_{ij}\right|.$$

11. How shall we compare different selections

- The ratios are only known with some inaccuracy.
- So, the resulting estimates of a_i are also inaccurate, i.e., they contain, in general, approximation error.
- In this talk, we will use two ways to compare the accuracy of different approaches:
  - by comparing the worst-case approximation error and
  - by comparing the mean squared approximation error.
- Now, we are ready to formulate the corresponding problem in precise terms.

12. Case When Experts Estimate the Ratios a_i/a_{i_0}

- Let us first consider the case when we ask experts to provide estimates w_{ii_0} for the ratios
$$\frac{a_i}{a_{i_0}}.$$
- In this case, we estimate a_i as $w_{ii_0} \cdot \widetilde{a}_{i_0}$.
- We have denoted the accuracy of estimating the ratio w_{ii_0} by ε.
- Let us analyze how this affects the accuracy of estimating a_i.
- For this purpose, let us denote the approximation error of approximating any quantity x with its approximate value $\widetilde{x}$ by $\Delta x \stackrel{\text{def}}{=} \widetilde{x} - x$.
- For a_i, the exact value – in the new measuring unit – is
$$a_i = \frac{a_i}{a_{i_0}} \cdot \widetilde{a}_{i_0}.$$
- Our estimate of this value is equal to $\widetilde{a}_i = w_{ii_0} \cdot \widetilde{a}_{i_0}$.

13. Case When Experts Estimate the Ratios a_i/a_{i_0} (cont-d)

- Thus, the approximation error $\Delta a_i \stackrel{\text{def}}{=} \widetilde{a}_i - a_i$ is equal to

$$\Delta a_i = \left(w_{ii_0} - \frac{a_i}{a_{i_0}}\right) \cdot \widetilde{a}_{i_0} = \Delta w_{ii_0} \cdot \widetilde{a}_{i_0}.$$

- Here we denoted $\Delta w_{ii_0} \stackrel{\text{def}}{=} w_{ii_0} - \dfrac{a_i}{a_{i_0}}$.
- So, the desired approximation error Δa_i of estimating a_i is obtained:
  - from the approximation error Δw_{ii_0} of estimating the corresponding ratio
  - by multiplying it by $\widetilde{a}_{i_0}$.

14. Case When Experts Estimate the Ratios a_i/a_{i_0} (cont-d)

- Thus:
  - whether we talk about the accuracy as the mean squared approximation error or the largest possible approximation error,
  - the accuracy δ_i with which we estimate a_i can be obtained
    - from the accuracy ε of estimating w_{ii_0}
    - by multiplying it by the same the same number $\widetilde{a}_{i_0}$:

$$\delta_i = \varepsilon \cdot \widetilde{a}_{i_0}.$$

15. Worst-case approach

- In the worst-case approach, we minimize the worst-case approximation error, i.e., we minimize the quantity

$$\delta \stackrel{\text{def}}{=} \max_{i \neq i_0} \delta_i = \varepsilon \cdot \widetilde{a}_{i_0}.$$

- Thus, to minimize this approximation error, we need to select, as the reference object i_0, the object with the *smallest* possible value of a_i.
- This explains one of the choices that turned out to be empirically successful.

16. Mean-squared approach

- In the mean-squared approach, we minimize the mean-squared approximation error, i.e., we minimize the quantity

$$\delta \stackrel{\text{def}}{=} \sqrt{\frac{1}{n-1} \cdot \sum_{i \neq i_0} \delta_i^2} = \sqrt{\frac{1}{n-1} \cdot \sum_{i \neq i_0} (\varepsilon \cdot \widetilde{a}_{i_0})^2} = \varepsilon \cdot \widetilde{a}_{i_0}.$$

- This is the exact same expression as in the worst-case approach.
- So, to minimize this approximation error:
  - we also need to select, as the reference object i_0,
  - the object with the *smallest* possible value of a_i.
- This is exactly one of the choices that turned out to be empirically successful.

17. Case When Experts Estimate the Ratios a_{i_0}/a_i

- Let us now consider the case when we ask experts to provide estimates $w_{i_0 i}$ for the ratios $\dfrac{a_{i_0}}{a_i}$.
- In this case, we estimate a_i as $w_{ii_0}^{-1} \cdot \widetilde{a}_{i_0}$.
- We have denoted the accuracy of estimating the ratio w_{ii_0} by ε.
- Let us analyze how this affect the accuracy of estimating a_i.
- In general, suppose that we approximate a quantity x by a value $\widetilde{x}$, with approximation error $\Delta x \stackrel{\text{def}}{=} \widetilde{x} - x$.
- Then, we have $x = \widetilde{x} - \Delta x$.
- We use this estimate to estimate the value $y = f(x)$ for a given function $f(x)$.
- In this case, our estimate $\widetilde{y}$ for y is obtained by plugging in the approximate value $\widetilde{x}$ into the formula $y = f(x)$, i.e., $\widetilde{y} = f(\widetilde{x})$.

18. Case When Experts Estimate the Ratios a_{i_0}/a_i (cont-d)

- Thus, the approximation error Δy of estimating y is equal to

$$\Delta y = \widetilde{y} - y = f(\widetilde{x}) - f(x) = f(\widetilde{x}) - f(\widetilde{x} - \Delta x).$$

- Approximation errors are usually small.
- So, the terms which are quadratic or higher order in terms of these errors can be safely ignored.
- For example, for the accuracy of 20%, the square is 4% which is much smaller.
- So, we expand the right-hand side of the above expression for Δy in Taylor series and safely ignore quadratic and higher order terms.
- This leaves only linear terms in this expansion.

19. Case When Experts Estimate the Ratios a_{i_0}/a_i (cont-d)

- As a result, we get $\Delta y = f'(\widetilde{x}) \cdot \Delta x$; here $f'(x)$, as usual, means the derivative.
  - Whether we look for the largest possible absolute value of Δy or for its mean-squared value,
  - this value can be obtained by multiplying the accuracy of approximating x by $|f'(x)|$.
- In our case, we have $x = w_{i_0 i}$ and $f(x) = x^{-1} \cdot \widetilde{a}_{i_0}$, thus
$$f'(x) = -x^{-2} \cdot \widetilde{a}_{i_0}.$$
- So, the accuracy δ_i with which we approximate a_i is equal to
$$\delta_i = w_{i_0 i}^{-2} \cdot \widetilde{a}_{i_0} \cdot \varepsilon.$$
- Here, $w_{i_0 i} \approx \dfrac{a_{i_0}}{a_i} \approx \dfrac{\widetilde{a}_{i_0}}{\widetilde{a}_i}$, so $w_{i_0 i}^{-2} \approx \left(\dfrac{\widetilde{a}_{i_0}}{\widetilde{a}_i}\right)^{-2} = \dfrac{(\widetilde{a}_i)^2}{(\widetilde{a}_{i_0})^2}$,
$$\delta_i \approx \frac{(\widetilde{a}_i)^2}{(\widetilde{a}_{i_0})^2} \cdot \widetilde{a}_{i_0} \cdot \varepsilon = \frac{(\widetilde{a}_i)^2}{\widetilde{a}_{i_0}} \cdot \varepsilon = (\widetilde{a}_i)^2 \cdot \frac{1}{\widetilde{a}_{i_0}} \cdot \varepsilon.$$

20. Worst-case approach

- In the worst-case approach, we minimize the worst-case approximation error, i.e., we minimize the quantity

$$\delta \stackrel{\text{def}}{=} \max_i \delta_i = \max_{i \ne i_0} (\widetilde{a}_i)^2 \cdot \frac{1}{\widetilde{a}_{i_0}} \cdot \varepsilon.$$

- Thus, to minimize this approximation error, we need to select, as the reference object i_0, the object with the *largest* possible value of a_i.
- This explains another of the two choices that turned out to be empirically successful.

21. Mean-squared approach

- In the mean-squared approach, we minimize the mean-squared approximation error, i.e., we minimize the quantity

$$\delta \stackrel{\text{def}}{=} \sqrt{\frac{1}{n-1} \cdot \sum_{i \neq i_0} (\widetilde{a}_i)^4} \cdot \frac{1}{\widetilde{a}_{i_0}} \cdot \varepsilon.$$

- To minimize this approximation error, we also need to select, as the reference object i_0, the object with the *largest* possible value of a_i.
- This is exactly one of the choices that turned out to be empirically successful.

22. Conclusions

- To accurate estimate the values of a quantity based on expert estimates, it is important to take into account that:
  - experts estimate the ratios of different values much more accurately
  - than they estimate the values themselves.
- It is therefore advisable to select one object, and to ask the expert to compare all other objects with the selected one.
- Empirical analysis shows that to achieve the best accuracy, we should select, as the reference object:
  - either the "best" object – i.e., the object with the largest value of the quantity of interest,
  - or the "worst" object, i.e., the object with the smallest value of this quantity.
- In this talk, we have provided a theoretical explanation for this empirical fact.

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