Why Best-Worst Method Works Well

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1. Formulation of the practical problem

- In many application areas, we rely on human estimates of different quantities.
- For example, when police investigates a crime, they rely on witnesses’ estimates of the suspect’s height and/or weight.
- In general:
  - we have $n$ objects, and
  - for each object $i, i = 1, \ldots, n$, we want to know the corresponding value $a_i$ of a quantity $a$.
- Estimates $\tilde{a}_i$ of untrained people are usually not very accurate – and thus, not very helpful.
- What we humans are much better at is *comparing* different values.
2. Formulation of the practical problem (cont-d)

- For example:
  - if we see two people, especially if we see them side by side,
  - then we can conclude that one of them is, e.g., 20% taller than the other.

- Similarly:
  - an instructor may not be able to accurately predict how exactly each student will perform on a test, but
  - usually, instructors can predict who will do better and who will do worse, and how better and how worse.

- So, for some pairs \((i, j)\), we ask the user to estimate the ratio \(\frac{a_i}{a_j}\) of the corresponding values.

- Based on these estimates, we want to reconstruct the values of the desired quantities.
3. Comment

- To get more accurate estimates, we can do two things.
- We can ask the same person several times to compare the same pairs of objects.
- Then, for each pair of objects, we get the average of the resulting estimate.
- We can also ask several persons and get an average of their estimates.
- This way of getting a more accurate estimate is known as the paradigm of crowd wisdom.
4. **Practical limitation**

- In general, the more information we have, the more accurate the resulting estimate.
- From this viewpoint, the more questions we ask about different pairs, the better.
- However, as the number of objects increases, the number of pairs increases quadratically, as \( \frac{n \cdot (n - 1)}{2} \sim n^2 \).
- For large \( n \), it becomes un-realistic to ask questions about all the pairs.
- With such possibility in mind, it is necessary to ask the smallest possible number of questions.
- A natural idea is:
  - to select one of the objects \( i_0 \), and
  - to only ask for ratios between this object and all other objects.
5. Empirical fact and what we do in this talk

- It has been empirically shown that to get the most accurate estimates, we need:
  - either to compare all the quantities with the smallest one,
  - or to compare all the quantities with the largest one.

- This is known as the *best-worst method*.

- In this talk, we provide a theoretical explanation for this empirical result.
6. What we mean by reconstructing the values $a_i$

- In order to formulate the problem in precise terms, let us first clarify what we mean by reconstructing the values $a_i$.

- Of course, if we only know the ratios, we cannot uniquely determine the actual values.

- Indeed:
  - if we multiply all the values $a_i$ by the same constant $c$,
  - then the ratios remain the same, while
  - the numerical values change.

- To avoid this non-uniqueness, a natural idea is to select some object $i_0$ for which we simply take $a_{i_0}^{\text{new}} \overset{\text{def}}{=} \tilde{a}_{i_0}$.

- This means, in effect, that we replace the original measuring unit with a new one, which is $\frac{a_{i_0}}{a_{i_0}}$ times smaller than the original measuring unit.
7. What we mean by reconstructing the values $a_i$ (cont-d)

- In terms of this unit, the new values $a_i^{\text{new}}$ of the desired quantity take the form

$$a_i^{\text{new}} = a_i \cdot \frac{\tilde{a}_{i0}}{a_{i0}}.$$ 

- Since we multiply all the values of the quantity by the same constant $c = \frac{a_i^{\text{new}}}{a_{i0}}$, the ratios remain the same:

$$\frac{a_i^{\text{new}}}{a_j^{\text{new}}} = \frac{a_i}{a_j}.$$ 

- A natural question is: which object $i_0$ should we select?
8. Once we selected $i_0$, how can we reconstruct the values $a_i$?

- If our estimates of the ratios were exact, then, in principle, by comparing all the objects with the selected object $i_0$, we could get the exact values of all the quantities $a_i$:
  
  - either as $a_i = \frac{a_i}{a_{i_0}} \cdot a_{i_0}$
  
  - or alternatively, as $a_i = \left(\frac{a_{i_0}}{a_i}\right)^{-1} \cdot a_{i_0}$.

- In practice, we do not know the exact ratios $\frac{a_i}{a_j}$, we only know the estimates $w_{ij}$ for these ratios: $w_{ij} \approx \frac{a_i}{a_j}$.

- So, by using these estimates instead of the actual ratios, we can provide estimates $\tilde{a}_i$ for the desired quantity by using:
  
  - either the formula $\tilde{a}_i = \tilde{a}_{i_0} \cdot w_{ii_0}$,
  
  - or, alternatively, the formula $\tilde{a}_i = \tilde{a}_{i_0} \cdot w_{i_0 i}^{-1}$. 
9. But is there a difference between these two approaches?

- At first glance, it may look like it does not matter what method we use.
- Indeed, the estimated ratios $\frac{a_i}{a_{i0}}$ and $\frac{a_{i0}}{a_i}$ are simply inverses to each other.
- So a consistent person should select estimates which are inverses as well, i.e., estimates for which
  \[ w_{ii0} = \frac{1}{w_{i0i}}. \]
- However, it is well known that people are not perfectly consistent.
- So, in general, these two estimates will lead to results:
  - which are not exactly mutually reverse and
  - which, thus, may lead to different estimates for the values $a_i$ of the desired quantity.
10. Need to take uncertainty into account

- In practice, as we have mentioned, we can only estimate the ratios with some accuracy.

- Let us denote the accuracy with which we estimate the ratios by $\varepsilon$.

- This can be the mean squared value of the difference between the actual ratio $\frac{a_i}{a_j}$ and our estimate $w_{ij}$.

- This corresponds, e.g., to the probabilistic approach to uncertainty.

- This can also be the largest possible absolute value of this difference $\left| \frac{a_i}{a_j} - w_{ij} \right|$. 
11. How shall we compare different selections

- The ratios are only known with some inaccuracy.
- So, the resulting estimates of $a_i$ are also inaccurate, i.e., they contain, in general, approximation error.
- In this talk, we will use two ways to compare the accuracy of different approaches:
  - by comparing the worst-case approximation error and
  - by comparing the mean squared approximation error.
- Now, we are ready to formulate the corresponding problem in precise terms.
12. Case When Experts Estimate the Ratios $a_i/a_{i_0}$

- Let us first consider the case when we ask experts to provide estimates $w_{ii_0}$ for the ratios $\frac{a_i}{a_{i_0}}$.

- In this case, we estimate $a_i$ as $w_{ii_0} \cdot \tilde{a}_{i_0}$.

- We have denoted the accuracy of estimating the ratio $w_{ii_0}$ by $\varepsilon$.

- Let us analyze how this affects the accuracy of estimating $a_i$.

- For this purpose, let us denote the approximation error of approximating any quantity $x$ with its approximate value $\tilde{x}$ by $\Delta x \overset{\text{def}}{=} \tilde{x} - x$.

- For $a_i$, the exact value – in the new measuring unit – is

  \[ a_i = \frac{a_i}{a_{i_0}} \cdot \tilde{a}_{i_0}. \]

- Our estimate of this value is equal to $\tilde{a}_i = w_{ii_0} \cdot \tilde{a}_{i_0}$. 
13. Case When Experts Estimate the Ratios $a_i/a_{i_0}$ (cont-d)

- Thus, the approximation error $\Delta a_i \overset{\text{def}}{=} \tilde{a}_i - a_i$ is equal to

$$\Delta a_i = \left( w_{ii_0} - \frac{a_i}{a_{i_0}} \right) \cdot \tilde{a}_{i_0} = \Delta w_{ii_0} \cdot \tilde{a}_{i_0}.$$ 

- Here we denoted $\Delta w_{ii_0} \overset{\text{def}}{=} w_{ii_0} - \frac{a_i}{a_{i_0}}$.

- So, the desired approximation error $\Delta a_i$ of estimating $a_i$ is obtained:
  - from the approximation error $\Delta w_{ii_0}$ of estimating the corresponding ratio
  - by multiplying it by $\tilde{a}_{i_0}$. 
Thus:

- whether we talk about the accuracy as the mean squared approximation error or the largest possible approximation error,
- the accuracy $\delta_i$ with which we estimate $a_i$ can be obtained
  * from the accuracy $\varepsilon$ of estimating $w_{ii_0}$
  * by multiplying it by the same the same number $\tilde{a}_{i_0}$:

$$\delta_i = \varepsilon \cdot \tilde{a}_{i_0}.$$
15. Worst-case approach

- In the worst-case approach, we minimize the worst-case approximation error, i.e., we minimize the quantity

\[ \delta \overset{\text{def}}{=} \max_{i \neq i_0} \delta_i = \varepsilon \cdot \tilde{a}_{i_0}. \]

- Thus, to minimize this approximation error, we need to select, as the reference object \( i_0 \), the object with the \textit{smallest} possible value of \( a_i \).

- This explains one of the choices that turned out to be empirically successful.
16. Mean-squared approach

- In the mean-squared approach, we minimize the mean-squared approximation error, i.e., we minimize the quantity

\[ \delta \overset{\text{def}}{=} \sqrt{\frac{1}{n-1} \cdot \sum_{i \neq i_0} \delta_i^2} = \sqrt{\frac{1}{n-1} \cdot \sum_{i \neq i_0} (\varepsilon \cdot \tilde{a}_{i_0})^2} = \varepsilon \cdot \tilde{a}_{i_0}. \]

- This is the exact same expression as in the worst-case approach.

- So, to minimize this approximation error:
  - we also need to select, as the reference object \( i_0 \),
  - the object with the \textit{smallest} possible value of \( a_i \).

- This is exactly one of the choices that turned out to be empirically successful.
17. Case When Experts Estimate the Ratios $a_{i_0}/a_i$

- Let us now consider the case when we ask experts to provide estimates $w_{i_0i}$ for the ratios $a_{i_0}/a_i$.
- In this case, we estimate $a_i$ as $w_{i0}^{-1} \cdot \tilde{a}_{i_0}$.
- We have denoted the accuracy of estimating the ratio $w_{ii_0}$ by $\varepsilon$.
- Let us analyze how this affect the accuracy of estimating $a_i$.
- In general, suppose that we approximate a quantity $x$ by a value $\tilde{x}$, with approximation error $\Delta x \overset{\text{def}}{=} \tilde{x} - x$.
- Then, we have $x = \tilde{x} - \Delta x$.
- We use this estimate to estimate the value $y = f(x)$ for a given function $f(x)$.
- In this case, our estimate $\tilde{y}$ for $y$ is obtained by plugging in the approximate value $\tilde{x}$ into the formula $y = f(x)$, i.e., $\tilde{y} = f(\tilde{x})$. 
Thus, the approximation error $\Delta y$ of estimating $y$ is equal to

$$\Delta y = \tilde{y} - y = f(\tilde{x}) - f(x) = f(\tilde{x}) - f(\tilde{x} - \Delta x).$$

Approximation errors are usually small.

So, the terms which are quadratic or higher order in terms of these errors can be safely ignored.

For example, for the accuracy of 20%, the square is 4% which is much smaller.

So, we expand the right-hand side of the above expression for $\Delta y$ in Taylor series and safely ignore quadratic and higher order terms.

This leaves only linear terms in this expansion.
As a result, we get $\Delta y = f'(\tilde{x}) \cdot \Delta x$; here $f'(x)$, as usual, means the derivative.

- Whether we look for the largest possible absolute value of $\Delta y$ or for its mean-squared value,
- this value can be obtained by multiplying the accuracy of approximating $x$ by $|f'(x)|$.

In our case, we have $x = w_{i_0i}$ and $f(x) = x^{-1} \cdot \tilde{a}_{i_0}$, thus

$$f'(x) = -x^{-2} \cdot \tilde{a}_{i_0}.$$

So, the accuracy $\delta_i$ with which we approximate $a_i$ is equal to

$$\delta_i = w_{i_0i}^{-2} \cdot \tilde{a}_{i_0} \cdot \varepsilon.$$

Here, $w_{i_0i} \approx \frac{a_{i_0}}{a_i} \approx \frac{\tilde{a}_{i_0}}{\tilde{a}_i}$, so $w_{i_0i}^{-2} \approx \left( \frac{\tilde{a}_{i_0}}{\tilde{a}_i} \right)^{-2} = \frac{(\tilde{a}_i)^2}{(\tilde{a}_{i_0})^2},$

$$\delta_i \approx \frac{(\tilde{a}_i)^2}{(\tilde{a}_{i_0})^2} \cdot \tilde{a}_{i_0} \cdot \varepsilon = \frac{(\tilde{a}_i)^2}{\tilde{a}_{i_0}} \cdot \varepsilon = (\tilde{a}_i)^2 \cdot \frac{1}{\tilde{a}_{i_0}} \cdot \varepsilon.$$
20. **Worst-case approach**

- In the worst-case approach, we minimize the worst-case approximation error, i.e., we minimize the quantity
  \[
  \delta \overset{\text{def}}{=} \max_i \delta_i = \max_{i \neq i_0} (\tilde{a}_i)^2 \cdot \frac{1}{\tilde{a}_{i_0}} \cdot \varepsilon.
  \]

- Thus, to minimize this approximation error, we need to select, as the reference object \(i_0\), the object with the *largest* possible value of \(a_i\).

- This explains another of the two choices that turned out to be empirically successful.
21. **Mean-squared approach**

- In the mean-squared approach, we minimize the mean-squared approximation error, i.e., we minimize the quantity

\[ \delta \overset{\text{def}}{=} \sqrt{\frac{1}{n-1} \cdot \sum_{i \neq i_0} (\tilde{a}_i)^4 \cdot \frac{1}{\tilde{a}_{i_0}} \cdot \varepsilon}. \]

- To minimize this approximation error, we also need to select, as the reference object \( i_0 \), the object with the *largest* possible value of \( a_i \).

- This is exactly one of the choices that turned out to be empirically successful.
22. Conclusions

- To accurately estimate the values of a quantity based on expert estimates, it is important to take into account that:
  - experts estimate the ratios of different values much more accurately
  - than they estimate the values themselves.
- It is therefore advisable to select one object, and to ask the expert to compare all other objects with the selected one.
- Empirical analysis shows that to achieve the best accuracy, we should select, as the reference object:
  - either the “best” object – i.e., the object with the largest value of the quantity of interest,
  - or the “worst” object, i.e., the object with the smallest value of this quantity.
- In this talk, we have provided a theoretical explanation for this empirical fact.
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