Do It Today Or Do It Tomorrow: Empirical Non-Exponential Discounting Explained by Symmetry Ideas

Francisco Zapata[1], Olga Kosheleva[1]
Vladik Kreinovich[1], Thongchai Dumrongpokaphan[2]

[1]University of Texas at El Paso, El Paso, Texas 79968, USA
fazg74@gmail.com, olgak@utep.edu, vladik@utep.edu
[2]Department of Mathematics, Chiang Mai University
Chiang Mai 50200 Thailand, tcd43@hotmial.com

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1. Discounting and Procrastination

- Future awards are less valuable than the same size awards given now.
- This phenomenon is known as *discounting*.
- Suppose that we have a task due by a deadline: e.g., submitting a grant proposal, or a paper to a conference.
- The reward is the same no matter when we finish this task – as long as it is before the deadline.
- Similarly, the negative effect caused by the need to do some boring stuff is the same no matter when we do it.
- The further in the future is this negative effect, the smaller is its influence on our today's happiness.
- Thus, a natural way to maximize today's happiness is to postpone this task as much as possible.
- This is exactly what people do.

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2. A Simple Theoretical Model of Discounting

- Suppose that 1 dollar tomorrow is equivalent to $D < 1$ dollars today.
- Thus, 1 dollar at the day $t+1$ is equivalent to D dollars in day t.
- In particular, 1 dollar in day $t_0 + 2$ is equivalent to D dollars at time $t_0 + 1$.
- But D dollars on day $t_0 + 1$ are equivalent to $D \cdot D$ dollars on day t_0.
- Thus, 1 dollar at day $t_0 + 2$ is equivalent to D^2 dollars at moment t_0.
- In general, 1 dollar at moment $t_0 + t$ is equivalent to $D(t) \stackrel{\text{def}}{=} D^t$ dollars at the current moment t_0.
- Here, $D(t) = D^t = \exp(-a \cdot t)$, where $a \stackrel{\text{def}}{=} -\ln(D)$; this discounting is thus known as *exponential*.

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3. Practical Problem with Exp Discounting

- For large t, $\exp(-a \cdot t)$ is indistinguishable from 0.
- So, a person looks for an immediate reward even if there is a negative downside in the distant future.
- Such behavior indeed happens:
  - a young man takes many loans without taking into account that in the future, he will have to pay;
  - a young person ruins his health by using drugs,
  - a person commits a crime ignoring that eventually, he will be caught and punished.
- Such behavior does happen, but such behavior is abnormal.
- This means that for most people, discounting decreases much slower than the exponential function.

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4. Discounting: Empirical Data.

- Empirical data shows: discounting indeed decreases much slower than predicted by the exp function.
- Namely, 1 dollar at moment $t_0 + t$ is equivalent to $D(t) = \dfrac{1}{1 + k \cdot t}$ dollars at moment t_0.
- This formula is known as *hyperbolic discounting*.
- In principle, there exist many functions that decrease slower than the exponential function $\exp(-a \cdot t)$.
- So why, out of all these functions, we observe the hyperbolic one?
- In this talk, we use symmetries to provide a theoretical explanation for the empirical discounting formula.

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5. Analysis of the Problem

- Let $D(t)$ denote the discounting of a reward which is t moments into the future.
- In other words, getting $D(t)$ dollars now is equivalent to getting 1 dollar after time t.
- By definition, $D(0)$ means getting 1 dollar with no delay, so $D(0) = 1$.
- It is also reasonable to require that as the time period time t increases, the value of the reward goes to 0:

$$\lim_{t \to +\infty} D(t) = 0.$$

- It is also reasonable to require that a small change in t should lead to small changes in $D(t)$.
- So, $D(t)$ should be differentiable (smooth).

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6. Analysis of the Problem (cont-d)

- If we delay all the rewards by some time s, then each value $D(t)$ will be replaced by a smaller value $D(t+s)$.
- We can describe this replacement as $D(t+s) = F_s(D(t))$, where the function $F_s(x)$:
  - re-scales the original discount value $D(t)$
  - into the new discount value $D(t+s)$.
- For the exponential discounting, the re-scaling $F_s(x)$ is linear: $D(t+s) = C \cdot D(t)$, where $C \stackrel{\text{def}}{=} \exp(-a \cdot s)$.
- So, we have $F_s(x) = C \cdot x$.
- For the hyperbolic discounting, the corresponding re-scaling $F_s(x)$ is not linear.
- Which re-scaling should we select?

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7. Which Re-Scalings Are Reasonable?

- Of course, linear re-scalings should be reasonable.
- Also, intuitively, if a re-scaling is reasonable, then its inverse should also be reasonable.
- Similarly:
  - if two re-scalings are reasonable,
  - then applying them one after another should also lead to a reasonable re-scaling.
- So, the class of all reasonable re-scalings should be closed under inversion and composition: be a *group*.
- We want to be able to determine the transformation from this group based on finitely many experiments.
- In each experiment, we gain a finite number of values.
- So, after a finite number of experiments, we can only determine a finite number of parameters.

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8. Which Re-Scalings Are Reasonable (cont-d)

- Thus, we should be able:
  - to select an element of the desired transformation group
  - based on the values of finitely many parameters.
- So, the corresponding transformation group should be *finite-dimensional*.
- Summarizing: we want all the transformations $F_s(x)$ to belong to:
  - a finite-dimensional transformation group of functions of one variable
  - that contains all linear transformations.

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9. Which Re-Scalings Are Reasonable: Answer

- Known: the only finite-dimensional transformation groups that contain all linear transformations are:
  - the group of all linear transformations and
  - the group of all fractional-linear transformations

$$\frac{a + b \cdot x}{1 + c \cdot x}.$$

- Thus, each reasonable re-scaling is fractionally linear:

$$D(t + s) = \frac{a(s) + b(s) \cdot D(t)}{1 + c(s) \cdot D(t)}.$$

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10. Main Result

- We say that a smooth decreasing function $D(t)$ is a *reasonable discounting function* if:
  - we have $D(0) = 1$ and $\lim\limits_{t\to\infty} D(t) = 0$, and
  - for every s, there exist values $a(s)$, $b(s)$, and $c(s)$ for which

$$D(t+s) = \frac{a(s) + b(s)\cdot D(t)}{1 + c(s)\cdot D(t)}.$$

Proposition. *$D(t)$ is a reasonable discounting function if and only if it has one of the following forms:*

$$D(t) = \exp(-a\cdot t), \quad D(t) = \frac{1}{1+k\cdot t},$$

$$D(t) = \frac{1+a}{1+a\cdot\exp(k\cdot t)}, \quad D(t) = \frac{a}{(a+1)\cdot\exp(k\cdot t) - 1}.$$

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11. Discussion

- The function $D(t) = \exp(-a \cdot t)$ is exponential discounting.
- The function $D(t) = \dfrac{1}{1 + k \cdot t}$ is hyperbolic discounting.
- Thus, we have found a theoretical justification for hyperbolic discounting.
- *Comments:*
  - The other two functions correspond to the more general case.
  - Both exp and hyperbolic discounting can be viewed as the limit case of these general formulas.

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13. Proof

- It is easy to show that each of the four functions $D(t)$ is a reasonable discounting function.
- Let us assume that $D(t)$ is a reasonable discounting function.
- Then, there exists functions $a(s)$, $b(s)$, and $c(s)$ for which, for all t and s, we have

$$D(t+s) = \frac{a(s) + b(s) \cdot D(t)}{1 + c(s) \cdot D(t)}.$$

- Let us first prove that $a(s) = 0$.
- Indeed, in the limit when $t \to \infty$, we have $D(t) \to 0$ and $D(t+s) \to 0$.
- Tending both sides of the above equality to the limit, we conclude that $a(s) = 0$.

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14. Proof (cont-d)

- Since $a(s) = 0$, we have $D(t+s) = \dfrac{b(s) \cdot D(t)}{1 + c(s) \cdot D(t)}$.
- Let us now prove that the functions $b(s)$ and $c(s)$ are differentiable.
- By definition of a reasonable discounting function, $D(t)$ is differentiable.
- If we multiply both sides of the above formula by the denominator, we get:

$$D(t+s) + c(s) \cdot D(t) \cdot D(t+s) = b(s) \cdot D(t), \text{ hence}$$

$$b(s) \cdot (-D(t)) + c(s) \cdot (D(t) \cdot D(t+s)) = -D(t+s).$$

- Thus, for each s, if we take $t_1 \neq t_2$, we get a system of two linear equations for the unknowns $b(s)$ and $c(s)$:

$$b(s) \cdot (-D(t_1)) + c(s) \cdot (D(t_1) \cdot D(t_1+s)) = -D(t_1+s);$$

$$b(s) \cdot (-D(t_2)) + c(s) \cdot (D(t_2) \cdot D(t_2+s)) = -D(t_2+s).$$

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15. Proof (cont-d)

- The solution of a system of linear equation can be explicitly described by the Cramer's rule.
- According to this rule, we have a differentiable formula that describes:
  - the solution to a system
  - in terms of the coefficients at the unknowns and of the right-hand sides.
- In our case, both coefficients and right-hand sides are differentiable functions of s.
- Thus, we conclude that the functions $b(s)$ and $c(s)$ are also differentiable.

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16. Proof (final)

- We know that $D(t+s) = \dfrac{b(s)\cdot D(t)}{1+c(s)\cdot D(t)}$.
- Let us differentiate both sides of this formula with respect to s and take $s=0$:

$$\frac{dD}{dt} = B\cdot D - C\cdot D^2, \text{ where } B \stackrel{\text{def}}{=} b'(0), C \stackrel{\text{def}}{=} c'(0).$$

- Thus, we get $\dfrac{dD}{B\cdot D - C\cdot D^2} = dt$, hence

$$\int \frac{dD}{B\cdot D - C\cdot D^2} = t + \text{const}.$$

- Integrating and taking into account monotonicity and $\lim\limits_{t\to\infty} D(t) = 0$, we get the desired functions.

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