Towards Parallel Quantum Computing: Standard Quantum Teleportation Algorithm Is, in Some Reasonable Sense, Unique

Oscar Galindo, Olga Kosheleva, and Vladik Kreinovich

University of Texas at El Paso, El Paso, Texas 79968, USA,
ogalindomo@miners.utep.edu, olgak@utep.edu, vladik@utep.edu

1. Need for Fast Data Processing

- Computational models can predict, with high probability, where a tornado will turn in the next hour.
- However, even on high performance computers, the computations take longer than an hour.
- This defeats the whole purpose of prediction.
- This lengthy computation time is caused by uncertainty:
  - if we had full information about the state of the atmosphere,
  - we could simply solve the corresponding system of partial differential equations.
- However, in practice, we have only partial information, we have uncertainty.

2. Need for Fast Data Processing (cont-d)

- Thus, to make reasonable predictions, we need to:
  - generate many different solutions, corresponding to several different situations – and
  - make predictions based on the frequency of solutions corresponding to different directions.
- This is a general phenomenon:
  - taking uncertainty into account drastically increases the computation time,
  - because, under uncertainty, we need to process several alternative scenarios instead of a single one.

3. Faster Processing Means Smaller Memory And Computation Cells

- One of the main limits on the computation speed is that all velocities are limited by the speed of light.
- The light is fast – it travels at 300 000 km/sec.
- However, for a current computer of size 30 cm, this means that
  - the fastest we can move information from one side of the computer to another is 1 nanosecond,
  - and even the simplest current computers have processing speed of several Gigahertz,
  - which means that several computation cycles take place while the information is transmitted.
- We thus need smaller computers – i.e., smaller memory and computation cells.

4. Need to Take Quantum Effects into Account

- Already in the existing computers, a memory cell sometimes consists of several dozen atoms.
- For such small objects, we need to take into account the laws of quantum physics.
- The main difference from traditional physics is in our ability to measure things.
- Indeed, the only way to measure a physical quantity is to interact with the corresponding object; e.g.:
  - to measure a distance to a faraway object,
  - we can send a laser beam to this object and measure the time that it takes for this beam to come back.
- This is how we measure, e.g., the distance to the Moon.
- For macro-size objects, the corresponding probe can be very small, much smaller than the object itself.

5. Quantum Effects (cont-d)

- Thus, we can safely ignore the effect of this probe on our object.
- We conclude that after the measurement, the object remains the same.
- We do not expect the distance to the Moon to change just because we hit the Moon with a laser beam.
- We can thus measure the Moon's location, velocity, and other characteristics with very high accuracy.
- We can also send a photon to a proton and measure its bouncing back.
- However, this photon is already of approximately the same size as the particle whose location we measure.
- As a result, every measurement changes the state of the particle.

6. Quantum Effects (cont-d)

- So, even if we get the particle's location, its speed changes.
- If we afterwards measure its speed, it will be different from the speed of the original particle.
- Because of this, for a micro-object, we cannot uniquely determine its state.
- We can only describe the probability of different measurement results.

7. Computer Scientists Managed to Transform This Lemon into Lemonade

- At first glance, this makes computations more complicated.
- As we decrease the size of the cells, we get quantum effects.
- So, the resulting states become only probabilistically predictable.
- In other words, we have a lot of noise added to our computations, noise that makes computations difficult.
- However, researchers managed to use quantum effects to speed up computations.
- This is done by re-arranging the corresponding computation schemes.

8. Lemon into Lemonade (cont-d)

- In non-quantum computing, finding an element in an unsorted database with n entries may require time n.
- Indeed, we may need to look at each record.
- In quantum computing, it is possible to find this element in much smaller time $\sqrt{n}$.
- An even larger speed-up is achieved in the problem of factorizing large integers.
- Traditional algorithms require time which is exponential in terms of the number's length.
- Thus, they are not feasible for large lengths.

9. Lemon into Lemonade (cont-d)

- However, quantum computing can do it in polynomial time; this application is important, since:
  - most current online encryption algorithms
  - are based on the difficulty of factoring large integers.
- So once quantum computers become a reality, we will be able to read all the so-far encrypted messages.

10. Need for Parallel Quantum Computing

- While quantum computing is fast, its speeds are also limited.
- To further speed up computations, a natural idea is to have several quantum computers working in parallel.
- Then each of them solves a part of the problem.
- This idea is similar to how we humans solve complex problems:
  - if a task is too difficult for one person to solve – be it building a big house or proving a theorem,
  - several people team up and together solve the task.

11. Need for Teleportation

- To successfully collaborate, quantum computers need to exchange intermediate states of their computations.
- Here lies a problem: for complex problems, we would like to use computers in different geographic areas.
- However, a quantum state gets changed when it is sent far away.
- Researchers have come up with a way to avoid this sending, called *teleportation*.
- There exists a scheme for teleportation.

12. Problem

- It is not clear how good is the current teleportation scheme.
- Maybe there are other schemes which are faster (or better in some other sense)?
- In this talk, we show that the existing teleportation scheme is, in some reasonable sense, unique.
- In this sense, this sense is the best.
- To explain this result, we start by a brief reminder of the basics of quantum physics.

13. Basic States in Quantum Physics

- In quantum physics:
  - in addition to the usual (non-quantum) states $s_1, s_2, \ldots,$
  - we also have *superpositions* of these states, i.e., states of the type $\alpha_1 \cdot s_1 + \alpha_2 \cdot s_2 + \ldots$
- Here $\alpha_1, \alpha_2, \ldots$ are complex numbers (called *amplitudes*) for which $|\alpha_1|^2 + |\alpha_2|^2 + \ldots = 1$.
- For example, a computer is formed from devices representing *binary digits* (*bits*, for short).
- These devices can be in two possible states: 0 and 1.
- In quantum physics, we also have superpositions

$$\alpha_0 \cdot |0\rangle + \alpha_1 \cdot |1\rangle, \text{ where } |\alpha_0|^2 + |\alpha_1|^2 = 1.$$

- The corresponding quantum system is known as a *quantum bit*, or *qubit*, for short.

14. Composite States in Quantum Physics

- There is a straightforward way to describe a composite system consisting of two independent subsystems.
- Due to independence, to describe the set of the system as a whole, it is sufficient to describe:
  - the state s of the first subsystem and
  - the state s' of the second subsystem.
- Thus, a state of the system as a whole is an ordered pair $\langle s, s' \rangle$ of the two states; let us denote:
  - possible states of the 1st subsystem by $s_1, s_2, \ldots;$
  - possible states of the 2nd subsystem by $s'_1, s'_2, \ldots$
- The subsystems are independent.
- So, the possible states of the 1st subsystem do not depend on the state of the 2nd.

15. Composite States (cont-d)

- Thus, the set of all states of the system as a whole is the set of all possible pairs $\langle s_i, s'_j\rangle$.
- The set of all such pairs is known as the *Cartesian product*; it is denoted by $\{s_1, s_2, \ldots\} \times \{s'_1, s'_2, \ldots\}$.
- These notations are usually simplified: e.g., $\langle 0, 1\rangle$ is denoted simply as 01.
- In quantum physics, we can also have superpositions of such states, i.e., the states of the type

$$\alpha_{11}\cdot\langle s_1, s'_1\rangle+\alpha_{12}\cdot\langle s_1, s'_2\rangle+\ldots+\alpha_{21}\cdot\langle s_2, s'_1\rangle+\alpha_{22}\cdot\langle s_2, s'_2\rangle+\ldots$$

- Here, $|\alpha_{11}|^2 + |\alpha_{12}|^2 + \ldots + |\alpha_{21}|^2 + |\alpha_{22}|^2 + \ldots = 1$.
- To describe such a state, we need to known all the values α_{ij}.
- These values form a matrix – i.e., in mathematical terms, a *tensor*.

16. Composite States (cont-d)

- Because of this fact, the set of all such states is known as the *tensor product* $S \otimes S'$, where:
  - S is the set of all possible quantum states of the first subsystem and
  - S' is the set of all possible quantum states of the second subsystem.
- So, the pair $\langle s, s' \rangle$ is denoted by $s \otimes s'$ and called a *tensor product* of the states s and s':
  - if the first subsystem is in the state s_i and the second subsystem is in the state s'_j,
  - then the state of the system is $\langle s_i, s'_j \rangle = s_i \otimes s'_j$.
- If $s = \alpha_1 \cdot s_1 + \alpha_2 \cdot s_2 + \ldots$ and $s' = \alpha'_1 \cdot s'_1 + \alpha'_2 \cdot s'_2 + \ldots$, then $s \oplus s' = \sum\limits_{i,j} \alpha_i \cdot \alpha'_j \cdot s_i \odot s'_j$.

17. Transformations in Quantum Physics

- Physically possible transformation are the mappings from state to state that satisfy the following properties:
  - superpositions get transformed into similar superpositions:

    $$T(\alpha_1 \cdot s_1 + \alpha_2 \cdots s_2 + \ldots) = \alpha_1 \cdot T(s_1) + \alpha_2 \cdot T(s_1) + \ldots,$$
  - $\sum |\alpha_i|^2 = 1$ is preserved: if $\sum |\alpha_i|^2 = 1$, then, for $T\left(\sum \alpha_i \cdot s_i\right) = \sum \beta_i \cdot s_i$, we have $\sum |\beta_i|^2 = 1$.
- Because of the first property, transformations are linear: $\sum \alpha_i \cdot s_i \to \sum \beta_i \cdot s_i$, with $\beta_i = \sum\limits_j t_{ij} \cdot \alpha_j$.
- Because of the second property, the matrix $T = (t_{ij})$ is *unitary*, i.e., $TT^\dagger = \mathbf{1}$, where $\mathbf{1}$ is a unit matrix.
- Here, $T^\dagger \stackrel{\text{def}}{=} (t^*_{ji})$, with z^* denoting the complex conjugate number $(a + b \cdot \mathrm{i})^* \stackrel{\text{def}}{=} a - b \cdot \mathrm{i}$.

18. Measurement Process in Quantum Physics

- For binary states $\alpha_0 \cdot |0\rangle + \alpha_1 \cdot |1\rangle$, if we want to measure whether the state is 0 or 1, then:
  - with probability $|\alpha_0|^2$, we get the result 0 – and the state turns into $|0\rangle$; and
  - with probability $|\alpha_1|^2$, we get the result 1 – and the state turns into $|1\rangle$.
- Since the result is either 0 or 1, the probabilities should add up to 1.
- This explains why physically possible states should satisfy the condition $|\alpha_0|^2 + |\alpha_1|^2 = 1$.
- In general, in a quantum state $\sum \alpha_i \cdot s_i$, we get s_i with probability $|\alpha_i|^2$.
- Once the measurement process detects the state s_i, the actual state turns into s_i.

19. Measurement Process (cont-d)

- Instead of the classical states s_i, we can use any orthonormal sequence of states $s'_i = \sum\limits_j t_{ij} \cdot s_j$:
  - for each i, we have $\|s'_i\|^2 = 1$, where $\|s'_i\|^2 \stackrel{\text{def}}{=} \sum\limits_j |t_{ij}|^2$ (*normal*), and
  - for each i and i', we have $s'_i \perp s'_{i'}$, i.e., $\langle s'_i | s'_{i'} \rangle = 0$, where $\langle s'_i | s'_{i'} \rangle \stackrel{\text{def}}{=} \sum\limits_j t_{ij} \cdot t^*_{i'j}$ (*orthogonal*).
- In a state $\sum \alpha'_i \cdot s'_i$, with probability $|\alpha'_i|^2$, the measurement result is s'_i and the state turns into s'_i.
- In general, instead of orthogonal vectors, we can have a sequence of orthogonal linear spaces $L_1, L_2, \ldots$
- Here $L_i \perp L_j$ means that $s_i \in L_i$ and $s_j \in L_j$ implies

$$s_i \perp s_j.$$

20. Measurement Process (cont-d)

- In this case, every state s can be represented as a sum $s = \sum s_i$ of the vectors $s_i \in L_i$.
- As a result of the measurement, with probability $\|s_i\|^2$:
  - we conclude that the state is in the space L_i, and
  - the original state turns into a new state $s_i/\|s_i\|$.

21. Need for Communication

- At one location, we have a particle in a certain state.
- We want to send this state to some other location.
- Usually, the sender is denoted by A and the receiver by B.
- In communications, it is common to call the sender Alice, and to call the receiver Bob:
  - states corresponding to Alice are usually described by using a subscript A, and
  - states corresponding to Bob are usually described by using a subscript B.

22. Communication Is Straightforward in Classical Physics

- In classical (pre-quantum) physics, the communication problem has a straightforward solution.
- If we want to communicate a state:
  - we measure all possible characteristics of this state,
  - send these values to Bob, and
  - let Bob reproduce the object with these characteristics.
- This is how, e.g., 3D printing works.
- This solution is based on the fact that:
  - in classical (non-quantum) physics
  - we can, in principle, measure all characteristic of a system without changing it.

23. Communication Is a Challenge in Quantum Physics

- The problem is that in quantum physics, such a straightforward approach is not possible.
- In quantum physics, every measurement changes the state.
- Moreover, each measurement irreversibly deletes some information about the state.
- For example, if we start with a state $\alpha_0 \cdot |0\rangle + \alpha_1 \cdot |1\rangle$, all we get after the measurement is either 0 or 1.
- There is no way to reconstruct the values α_0 and α_1 that characterize the original state.
- Since we cannot use a direct approach for communicating a state, we need to use an indirect approach.
- This approach is known as *teleportation*.

24. What We Consider in This Talk

- We consider the quantum analogue of the simplest possible non-quantum state.
- The simplest case when communication is needed is when the system can be in two different states.
- In the computer, such situation can be naturally described if we associate these states with 0 and 1.
- Alice has a state $\alpha_0 \cdot |0\rangle + \alpha_1 \cdot |1\rangle$ that she wants to communicate to Bob.
- The above state is not exclusively Alice's or Bob's.
- So, to describe this state, we will use the next letter C.
- In these terms, Alice has a state $\alpha_0 \cdot |0\rangle_C + \alpha_1 \cdot |1\rangle_C$.
- She wants to communicate this state to Bob.

25. Preparing for Teleportation: an Entangled State

- To make teleportation possible, Alice and Bob prepare a special *entangled* state:

$$\frac{1}{\sqrt{2}} \cdot |0_A 1_B\rangle + \frac{1}{\sqrt{2}} \cdot |1_A 0_B\rangle.$$

- This state is a superposition of two classical states:
  - the state $0_A 1_B$ in which A is in state 0 and B is in state 1, and
  - the state $1_A 0_B$ in which A is in state 1 and B is in state 0.
- At first, the state C is independent of A and B.
- So, the joint state is a tensor product of the AB-state and the C-state:

$$\frac{\alpha_0}{\sqrt{2}} \cdot |0_A 1_B 0_C\rangle + \frac{\alpha_1}{\sqrt{2}} \cdot |0_A 1_B 1_C\rangle + \frac{\alpha_0}{\sqrt{2}} \cdot |1_A 0_B 0_C\rangle + \frac{\alpha_1}{\sqrt{2}} \cdot |1_A 0_B 1_C\rangle.$$

26. First Stage: Measurement

- First, Alice performs a measurement procedure on the parts A and C which are available to her.
- We perform the measurement w.r.t. $L_i = L_B \otimes t_i$.
- Here, L_B is the set of all possible linear combinations of $|0\rangle_B$ and $|1\rangle_B$.
- The states t_i are as follows:

$$t_1 = \frac{1}{\sqrt{2}} \cdot |0_A 0_C\rangle + \frac{1}{\sqrt{2}} \cdot |1_A 1_C\rangle;$$

$$t_2 = \frac{1}{\sqrt{2}} \cdot |0_A 0_C\rangle - \frac{1}{\sqrt{2}} \cdot |1_A 1_C\rangle;$$

$$t_3 = \frac{1}{\sqrt{2}} \cdot |0_A 1_C\rangle + \frac{1}{\sqrt{2}} \cdot |1_A 0_C\rangle;$$

$$t_4 = \frac{1}{\sqrt{2}} \cdot |0_A 1_C\rangle - \frac{1}{\sqrt{2}} \cdot |1_A 0_C\rangle.$$

27. First Stage: Measurement (cont-d)

- One can easily check that the states t_i are orthonormal, hence the spaces L_i are orthogonal.
- Let's represent the state in as $s = \sum s_i$, with $s_i \in L_i$:

$$s_1 = \left(\frac{\alpha_0}{2} \cdot |1_B\rangle + \frac{\alpha_1}{2}|0_B\rangle\right) \otimes t_1,$$

$$s_2 = \left(\frac{\alpha_0}{2} \cdot |1_B\rangle - \frac{\alpha_1}{2} \cdot |0_B\rangle\right) \otimes t_2,$$

$$s_3 = \left(\frac{\alpha_1}{2} \cdot |1_B\rangle + \frac{\alpha_0}{2} \cdot |0_B\rangle\right) \otimes t_3,$$

$$s_4 = \left(\frac{\alpha_1}{2} \cdot |1_B\rangle - \frac{\alpha_0}{2} \cdot |0_B\rangle\right) \otimes t_4.$$

- Here, for each i, we have $\|s_i\| = \dfrac{1}{2}$.

28. First Stage: Measurement (cont-d)

- So, with equal probability of $\dfrac{1}{4}$, we get one of the following four states – and Alice knows which one it is:

$$(\alpha_0 \cdot |1_B\rangle + \alpha_1 \cdot |0_B\rangle) \otimes t_1;$$

$$(\alpha_0 \cdot |1_B\rangle - \alpha_1 \cdot |0_B\rangle) \otimes t_2;$$

$$(\alpha_1 \cdot |1_B\rangle + \alpha_0 \cdot |0_B\rangle) \otimes t_3;$$

$$(\alpha_1 \cdot |1_B\rangle - \alpha_0 \cdot |0_B\rangle) \otimes t_4.$$

29. Two Final Stages

- Alice sends to Bob the measurement result.
- So, Bob knows in which the four states the system is.
- Bob performs a transformation of his state B.
- In the first case, he uses a unitary transformation that swaps $|0\rangle_B$ and $|1\rangle_B$: $t_{01} = t_{10} = 1$ and $t_{00} = t_{11} = 0$.
- In the second case, he uses a unitary transformation for which $t_{01} = 1$, $t_{10} = -1$ and $t_{00} = t_{11} = 0$.
- In the third case, he already has the desired state.
- In the fourth case, he uses a unitary transformation for which $t_{00} = -1$, $t_{11} = 1$, and $t_{01} = t_{10} = 0$.
- As a result, in all fours cases, he gets the original state $\alpha_0 \cdot |0\rangle_B + \alpha_1 \cdot |1\rangle_B$.

30. Formulation of the Problem

- Teleportation is possible because we have prepared an *entangled* state.
- This is a state s_{AB} in which the states of Alice and Bob are not independent.
- However, the above is not the only possible entangled state.
- Let us consider, instead, a general joint state of two qubits:

$$a_{00} \cdot |0_A 0_B\rangle + a_{01} \cdot |0_A 1_B\rangle + a_{10} \cdot |1_A 0_B\rangle + a_{11} \cdot |1_A 1_B\rangle.$$

- What will happen if we use this more general entangled state?

31. Analysis of the Problem

- For the general state, the joint state of all three sub-systems has the form

$$\alpha_0 \cdot a_{00} \cdot |0_A 0_B 0_C\rangle + \alpha_1 \cdot a_{00} \cdot |0_A 0_B 1_C\rangle +$$
$$\alpha_0 \cdot a_{01} \cdot |0_A 1_B 0_C\rangle + \alpha_1 \cdot a_{01} \cdot |0_A 1_B 1_C\rangle +$$
$$\alpha_0 \cdot a_{10} \cdot |1_A 0_B 0_C\rangle + \alpha_1 \cdot a_{10} \cdot |1_A 0_B 1_C\rangle +$$
$$\alpha_0 \cdot a_{11} \cdot |1_A 1_B 0_C\rangle + \alpha_1 \cdot a_{11} \cdot |1_A 1_B 1_C\rangle.$$

- Substituting expressions for s_i, we get $s = S_1 \otimes t_1 + S_2 \otimes t_2 + \ldots$, where:

$$S_1 = \left(\frac{\alpha_0 \cdot a_{00}}{\sqrt{2}} + \frac{\alpha_1 \cdot a_{10}}{\sqrt{2}}\right) \cdot |0\rangle_B + \left(\frac{\alpha_0 \cdot a_{01}}{\sqrt{2}} + \frac{\alpha_1 \cdot a_{11}}{\sqrt{2}}\right) \cdot |1\rangle_B.$$

- $S_2, \ldots$ are described by similar expressions.
- This means that after the measurement, Bob will have the normalized state $S_1/\|S_1\|$.

32. Analysis of the Problem (cont-d)

- To perform teleportation, we need to transform this state into the original state $\alpha_0 \cdot |0\rangle_B + \alpha_1 \cdot |1\rangle_B$.
- Thus, the transformation from the resulting state $S_1/\|S_1\|$ to the original state must be unitary.
- It is known that the inverse transformation to a unitary one is also unitary.
- In general, a unitary transformation transforms orthonormal states into orthonormal ones.

33. Analysis of the Problem (cont-d)

- So, the inverse transformation:
  - maps the state $|0\rangle_B$ (corresponding to $\alpha_0 = 1$ and $\alpha_1 = 0$) into a new state
  $$|1'\rangle_B \stackrel{\text{def}}{=} \text{const} \cdot (a_{00} \cdot |0\rangle_B + a_{01} \cdot |1\rangle_B),$$
  - maps the state $|1\rangle_B$ (corresponding to $\alpha_0 = 0$ and $\alpha_1 = 1$) into a new state
  $$|0'\rangle_B \stackrel{\text{def}}{=} \text{const} \cdot (a_{10} \cdot |0\rangle_B + a_{11} \cdot |1\rangle_B).$$
- It should transform two original orthonormal vectors $|0\rangle_B$, $|1\rangle_B$ into two new orthonormal ones $|0'\rangle_B$, $|1'\rangle_B$.
- In terms of these new states, the entangled state is
$$\text{const} \cdot (|0\rangle_A \otimes |1'\rangle_B + |1\rangle_B \otimes |0'\rangle_B).$$
- The sum of the squares of absolute values of all the coefficients should add up to 1.

34. Analysis of the Problem (cont-d)

- Then const $= \dfrac{1}{\sqrt{2}}$, and the entangled state takes the familiar form $\dfrac{1}{\sqrt{2}} \cdot (|0\rangle_A \otimes |1'\rangle_B + |1\rangle_B \otimes |0'\rangle_B)$.
- This is exactly the entangled state used in the standard teleportation algorithm.

35. Conclusion

- From the technical viewpoint:
  - the only entangled state that leads to a successful teleportation
  - is the state corresponding to the standard quantum teleportation algorithm,
  - for some orthornomal states $|0'\rangle_B$ and $|1'\rangle_B$.
- Thus, we have shown that, indeed, the existing quantum teleportation algorithm is unique.
- So we should not waste our time and effort looking for more efficient alternative teleportation algorithms.

36. Acknowledgments

- This work was supported in part by the US National Science Foundation grant HRD-1242122 (Cyber-ShARE).