The Notion of Utility

How to Estimate . . .

How to Estimate . . .

From Utility to . . .

From Interval-Valued . . .

Decisions under . . .

Nash's Bargaining . . .

Group Decision . . .

Paradox of Love

Decision Making under Interval and Fuzzy Uncertainty: Utility Approach

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Home Page

Title Page

◀◀ ▶▶

◀ ▶

Page 1 of 62

Go Back

Full Screen

Close

Quit

The Notion of Utility

How to Estimate . . .

How to Estimate . . .

From Utility to . . .

From Interval-Valued . . .

Decisions under . . .

Nash's Bargaining . . .

Group Decision . . .

Paradox of Love

1. Decision Making: General Need and Traditional Approach

- To make a decision, we must:
  - find out the user's preference, and
  - help the user select an alternative which is the best – according to these preferences.

- Traditional approach is based on an assumption that for each two alternatives A' and A'', a user can tell:
  - whether the first alternative is better for him/her; we will denote this by $A'' < A'$;
  - or the second alternative is better; we will denote this by $A' < A''$;
  - or the two given alternatives are of equal value to the user; we will denote this by $A' = A''$.

The Notion of Utility

How to Estimate . . .

How to Estimate . . .

From Utility to . . .

From Interval-Valued . . .

Decisions under . . .

Nash's Bargaining . . .

Group Decision . . .

Paradox of Love

2. The Notion of Utility

- Under the above assumption, we can form a natural numerical scale for describing preferences.

- Let us select a very bad alternative A_0 and a very good alternative A_1.

- Then, most other alternatives are better than A_0 but worse than A_1.

- For every prob. $p \in [0,1]$, we can form a lottery $L(p)$ in which we get A_1 w/prob. p and A_0 w/prob. $1 - p$.

- When $p = 0$, this lottery simply coincides with the alternative A_0: $L(0) = A_0$.

- The larger the probability p of the positive outcome increases, the better the result:

$$p' < p'' \text{ implies } L(p') < L(p'').$$

The Notion of Utility

How to Estimate . . .

How to Estimate . . .

From Utility to . . .

From Interval-Valued . . .

Decisions under . . .

Nash's Bargaining . . .

Group Decision . . .

Paradox of Love

3.   The Notion of Utility (cont-d)

- Finally, for $p = 1$, the lottery coincides with the alternative A_1: $L(1) = A_1$.

- Thus, we have a continuous scale of alternatives $L(p)$ that monotonically goes from $L(0) = A_0$ to $L(1) = A_1$.

- Due to monotonicity, when p increases, we first have $L(p) < A$, then we have $L(p) > A$.

- The threshold value is called the *utility* of the alternative A:

$$u(A) \stackrel{\text{def}}{=} \sup\{p : L(p) < A\} = \inf\{p : L(p) > A\}.$$

- Then, for every $\varepsilon > 0$, we have

$$L(u(A) - \varepsilon) < A < L(u(A) + \varepsilon).$$

- We will describe such (almost) equivalence by $\equiv$, i.e., we will write that $A \equiv L(u(A))$.

4. Fast Iterative Process for Determining $u(A)$

- *Initially:* we know the values $\underline{u} = 0$ and $\overline{u} = 1$ such that $A \equiv L(u(A))$ for some $u(A) \in [\underline{u}, \overline{u}]$.

- *What we do:* we compute the midpoint u_{mid} of the interval $[\underline{u}, \overline{u}]$ and compare A with $L(u_{\mathrm{mid}})$.

- *Possibilities:* $A \leq L(u_{\mathrm{mid}})$ and $L(u_{\mathrm{mid}}) \leq A$.

- *Case 1:* if $A \leq L(u_{\mathrm{mid}})$, then $u(A) \leq u_{\mathrm{mid}}$, so

$$u \in [\underline{u}, u_{\mathrm{mid}}].$$

- *Case 2:* if $L(u_{\mathrm{mid}}) \leq A$, then $u_{\mathrm{mid}} \leq u(A)$, so

$$u \in [u_{\mathrm{mid}}, \overline{u}].$$

- After each iteration, we decrease the width of the interval $[\underline{u}, \overline{u}]$ by half.

- After k iterations, we get an interval of width 2^{-k} which contains $u(A)$ – i.e., we get $u(A)$ w/accuracy 2^{-k}.

The Notion of Utility

How to Estimate . . .

How to Estimate . . .

From Utility to . . .

From Interval-Valued . . .

Decisions under . . .

Nash's Bargaining . . .

Group Decision . . .

Paradox of Love

The Notion of Utility

How to Estimate...

How to Estimate...

From Utility to...

From Interval-Valued...

Decisions under...

Nash's Bargaining...

Group Decision...

Paradox of Love

5. How to Make a Decision Based on Utility Values

- Suppose that we have found the utilities $u(A')$, $u(A'')$, ..., of the alternatives A', A'', ...

- Which of these alternatives should we choose?

- By definition of utility, we have:

  - $A \equiv L(u(A))$ for every alternative A, and
  - $L(p') < L(p'')$ if and only if $p' < p''$.

- We can thus conclude that A' is preferable to A'' if and only if $u(A') > u(A'')$.

- In other words, we should always select an alternative with the largest possible value of utility.

- Interval techniques can help in finding the optimizing decision.

Home Page

Title Page

◀◀   ▶▶

◀   ▶

Page 6 of 62

Go Back

Full Screen

Close

Quit

6. How to Estimate Utility of an Action

- For each action, we usually know possible outcomes $S_1, \ldots, S_n$.

- We can often estimate the prob. $p_1, \ldots, p_n$ of these outcomes.

- By definition of utility, each situation S_i is equiv. to a lottery $L(u(S_i))$ in which we get:

  - A_1 with probability $u(S_i)$ and
  - A_0 with the remaining probability $1 - u(S_i)$.

- Thus, the action is equivalent to a complex lottery in which:

  - first, we select one of the situations S_i with probability p_i: $P(S_i) = p_i$;
  - then, depending on S_i, we get A_1 with probability $P(A_1 \mid S_i) = u(S_i)$ and A_0 w/probability $1 - u(S_i)$.

The Notion of Utility

How to Estimate . . .

How to Estimate . . .

From Utility to . . .

From Interval-Valued . . .

Decisions under . . .

Nash's Bargaining . . .

Group Decision . . .

Paradox of Love

7. How to Estimate Utility of an Action (cont-d)

- *Reminder:*
  - first, we select one of the situations S_i with probability p_i: $P(S_i) = p_i$;
  - then, depending on S_i, we get A_1 with probability $P(A_1 \mid S_i) = u(S_i)$ and A_0 w/probability $1 - u(S_i)$.

- The prob. of getting A_1 in this complex lottery is:

$$P(A_1) = \sum_{i=1}^{n} P(A_1 \mid S_i) \cdot P(S_i) = \sum_{i=1}^{n} u(S_i) \cdot p_i.$$

- In the complex lottery, we get:
  - A_1 with prob. $u = \sum_{i=1}^{n} p_i \cdot u(S_i)$, and
  - A_0 w/prob. $1 - u$.

- So, we should select the action with the largest value of expected utility $u = \sum p_i \cdot u(S_i)$.

The Notion of Utility

How to Estimate . . .

How to Estimate . . .

From Utility to . . .

From Interval-Valued . . .

Decisions under . . .

Nash's Bargaining . . .

Group Decision . . .

Paradox of Love

8.   Non-Uniqueness of Utility

- The above definition of utility u depends on A_0, A_1.

- What if we use different alternatives A_0' and A_1'?

- Every A is equivalent to a lottery $L(u(A))$ in which we get A_1 w/prob. $u(A)$ and A_0 w/prob. $1 - u(A)$.

- For simplicity, let us assume that $A_0' < A_0 < A_1 < A_1'$.

- Then, $A_0 \equiv L'(u'(A_0))$ and $A_1 \equiv L'(u'(A_1))$.

- So, A is equivalent to a complex lottery in which:

  1) we select A_1 w/prob. $u(A)$ and A_0 w/prob. $1 - u(A)$;
  2) depending on A_i, we get A_1' w/prob. $u'(A_i)$ and A_0' w/prob. $1 - u'(A_i)$.

- In this complex lottery, we get A_1' with probability $u'(A) = u(A) \cdot (u'(A_1) - u'(A_0)) + u'(A_0)$.

- So, in general, utility is defined modulo an (increasing) linear transformation $u' = a \cdot u + b$, with $a > 0$.

Home Page

Title Page

◀◀   ▶▶

◀   ▶

Page 9 of 62

Go Back

Full Screen

Close

Quit

9. Subjective Probabilities

- In practice, we often do not know the probabilities p_i of different outcomes.

- For each event E, a natural way to estimate its subjective probability is to fix a prize (e.g., \$1) and compare:

  - the lottery ℓ_E in which we get the fixed prize if the event E occurs and 0 is it does not occur, with

  - a lottery $\ell(p)$ in which we get the same amount with probability p.

- Here, similarly to the utility case, we get a value $ps(E)$ for which, for every $\varepsilon > 0$:

$$\ell(ps(E) - \varepsilon) < \ell_E < \ell(ps(E) + \varepsilon).$$

- Then, the utility of an action with possible outcomes $S_1, \ldots, S_n$ is equal to $u = \sum_{i=1}^{n} ps(E_i) \cdot u(S_i)$.

The Notion of Utility

How to Estimate . . .

How to Estimate . . .

From Utility to . . .

From Interval-Valued . . .

Decisions under . . .

Nash's Bargaining . . .

Group Decision . . .

Paradox of Love

The Notion of Utility

How to Estimate...

How to Estimate...

From Utility to...

From Interval-Valued...

Decisions under...

Nash's Bargaining...

Group Decision...

Paradox of Love

10. Beyond Traditional Decision Making: Towards a More Realistic Description

- Previously, we assumed that a user can always decide which of the two alternatives A' and A'' is better:

  - either $A' < A''$,
  - or $A'' < A'$,
  - or $A' \equiv A''$.

- In practice, a user is sometimes unable to meaningfully decide between the two alternatives; denoted $A' \parallel A''$.

- In mathematical terms, this means that the preference relation:

  - is no longer a *total* (linear) order,
  - it can be a *partial* order.

Home Page

Title Page

◀◀   ▶▶

◀   ▶

Page 11 of 62

Go Back

Full Screen

Close

Quit

11. From Utility to Interval-Valued Utility

- Similarly to the traditional decision making approach:

  - we select two alternatives $A_0 < A_1$ and

  - we compare each alternative A which is better than A_0 and worse than A_1 with lotteries $L(p)$.

- Since preference is a *partial* order, in general:

  $$\underline{u}(A) \stackrel{\text{def}}{=} \sup\{p : L(p) < A\} < \overline{u}(A) \stackrel{\text{def}}{=} \inf\{p : L(p) > A\}.$$

- For each alternative A, instead of a single value $u(A)$ of the utility, we now have an *interval* $[\underline{u}(A), \overline{u}(A)]$ s.t.:

  - if $p < \underline{u}(A)$, then $L(p) < A$;

  - if $p > \overline{u}(A)$, then $A < L(p)$; and

  - if $\underline{u}(A) < p < \overline{u}(A)$, then $A \parallel L(p)$.

- We will call this interval the *utility* of the alternative A.

The Notion of Utility

How to Estimate . . .

How to Estimate . . .

From Utility to . . .

From Interval-Valued . . .

Decisions under . . .

Nash's Bargaining . . .

Group Decision . . .

Paradox of Love

12.   From Interval-Valued to Fuzzy-Valued Utility

- The traditional approach to decision making assumes that for every two alternatives A and A':

$$\text{either } A < A' \text{ or } A' < A \text{ or } A \equiv A'.$$

- Interval approach allows the fourth alternative: $A \parallel A'$.

- In practice, for different pairs A and A', we have different degrees of confidence that $A < A'$.

- If we select $\alpha \in (0, 1)$, then we can form intervals corresponding to $A < A'$ with confidence $\geq \alpha$.

- Utility intervals corresponding to different degrees α form a *fuzzy* utility value.

13. Interval-Valued Utilities and Interval-Valued Subjective Probabilities

- To feasibly elicit the values $\underline{u}(A)$ and $\overline{u}(A)$, we:

  1) starting w/$[\underline{u}, \overline{u}] = [0, 1]$, bisect an interval s.t. $L(\underline{u}) < A < L(\overline{u})$ until we find u_0 s.t. $A \parallel L(u_0)$;

  2) by bisecting an interval $[\underline{u}, u_0]$ for which $L(\underline{u}) < A \parallel L(u_0)$, we find $\underline{u}(A)$;

  3) by bisecting an interval $[u_0, \overline{u}]$ for which $L(u_0) \parallel A < L(\overline{u})$, we find $\overline{u}(A)$.

- Similarly, when we estimate the probability of an event E:

  – we no longer get a single value $ps(E)$;

  – we get an *interval* $[\underline{ps}(E), \overline{ps}(E)]$ of possible values of probability.

- By using bisection, we can feasibly elicit the values $\underline{ps}(E)$ and $\overline{ps}(E)$.

The Notion of Utility

How to Estimate . . .

How to Estimate . . .

From Utility to . . .

From Interval-Valued . . .

Decisions under . . .

Nash's Bargaining . . .

Group Decision . . .

Paradox of Love

14. Decision Making Under Interval Uncertainty

- *Situation:* for each possible decision d, we know the interval $[\underline{u}(d), \overline{u}(d)]$ of possible values of utility.

- *Questions:* which decision shall we select?

- *Natural idea:* select all decisions d_0 that *may* be optimal, i.e., which are optimal for some function

$$u(d) \in [\underline{u}(d), \overline{u}(d)].$$

- *Problem:* checking all possible functions is not feasible.

- *Solution:* the above condition is equivalent to an easier-to-check one:

$$\overline{u}(d_0) \geq \max_d \underline{u}(d).$$

- *Interval computations* can help in describing the range of all such d_0.

- *Remaining problem:* in practice, we would like to select *one* decision; which one should be select?

The Notion of Utility

How to Estimate . . .

How to Estimate . . .

From Utility to . . .

From Interval-Valued . . .

Decisions under . . .

Nash's Bargaining . . .

Group Decision . . .

Paradox of Love

15.   Need for Definite Decision Making

- *At first glance:* if $A' \parallel A''$, it does not matter whether we recommend alternative A' or alternative A''.

- Let us show that this is *not* a good recommendation.

- E.g., let A be an alternative about which we know nothing, i.e., $[\underline{u}(A), \overline{u}(A)] = [0, 1]$.

- In this case, A is indistinguishable both from a "good" lottery $L(0.999)$ and a "bad" lottery $L(0.001)$.

- Suppose that we recommend, to the user, that A is equivalent both to $L(0.999)$ and to $L(0.001)$.

- Then this user will feel comfortable:
  - first, exchanging $L(0.999)$ with A, and
  - then, exchanging A with $L(0.001)$.

- So, following our recommendations, the user switches from a very good alternative to a very bad one.

The Notion of Utility

How to Estimate . . .

How to Estimate . . .

From Utility to . . .

From Interval-Valued . . .

Decisions under . . .

Nash's Bargaining . . .

Group Decision . . .

Paradox of Love

16. Need for Definite Decision Making (cont-d)

- The above argument does not depend on the fact that we assumed complete ignorance about A:

  - every time we recommend that the alternative A is "equivalent" both to $L(p)$ and to $L(p')$ ($p < p'$),

  - we make the user vulnerable to a similar switch from a better alternative $L(p')$ to a worse one $L(p)$.

- Thus, there should be only a single value p for which A can be reasonably exchanged with $L(p)$.

- In precise terms:

  - we start with the utility interval $[\underline{u}(A), \overline{u}(A)]$, and

  - we need to select a single $u(A)$ for which it is reasonable to exchange A with a lottery $L(u)$.

- How can we find this value $u(A)$?

17. Decisions under Interval Uncertainty: Hurwicz Optimism-Pessimism Criterion

- *Reminder:* we need to assign, to each interval $[\underline{u}, \overline{u}]$, a utility value $u(\underline{u}, \overline{u}) \in [\underline{u}, \overline{u}]$.

- *History:* this problem was first handled in 1951, by the future Nobelist Leonid Hurwicz.

- *Notation:* let us denote $\alpha_H \stackrel{\text{def}}{=} u(0, 1)$.

- *Reminder:* utility is determined modulo a linear transformation $u' = a \cdot u + b$.

- *Reasonable to require:* the equivalent utility does not change with re-scaling: for $a > 0$ and b,

$$u(a \cdot u^- + b, a \cdot u^+ + b) = a \cdot u(u^-, u^+) + b.$$

- For $u^- = 0$, $u^+ = 1$, $a = \overline{u} - \underline{u}$, and $b = \underline{u}$, we get

$$u(\underline{u}, \overline{u}) = \alpha_H \cdot (\overline{u} - \underline{u}) + \underline{u} = \alpha_H \cdot \overline{u} + (1 - \alpha_H) \cdot \underline{u}.$$

18.   Hurwicz Optimism-Pessimism Criterion (cont)

- The expression $\alpha_H \cdot \overline{u} + (1 - \alpha_H) \cdot \underline{u}$ is called *optimism-pessimism criterion*, because:

  - when $\alpha_H = 1$, we make a decision based on the most optimistic possible values $u = \overline{u}$;

  - when $\alpha_H = 0$, we make a decision based on the most pessimistic possible values $u = \underline{u}$;

  - for intermediate values $\alpha_H \in (0, 1)$, we take a weighted average of the optimistic and pessimistic values.

- According to this criterion:

  - if we have several alternatives A', . . . , with interval-valued utilities $[\underline{u}(A'), \overline{u}(A')]$, . . . ,

  - we recommend an alternative A that maximizes

$$\alpha_H \cdot \overline{u}(A) + (1 - \alpha_H) \cdot \underline{u}(A).$$

The Notion of Utility

How to Estimate . . .

How to Estimate . . .

From Utility to . . .

From Interval-Valued . . .

Decisions under . . .

Nash's Bargaining . . .

Group Decision . . .

Paradox of Love

19. Which Value α_H Should We Choose? An Argument in Favor of $\alpha_H = 0.5$

- Let us take an event E about which we know nothing.

- For a lottery L^+ in which we get A_1 if E and A_0 otherwise, the utility interval is $[0,1]$.

- Thus, the equiv. utility of L^+ is $\alpha_H \cdot 1 + (1-\alpha_H) \cdot 0 = \alpha_H$.

- For a lottery L^- in which we get A_0 if E and A_1 otherwise, the utility is $[0,1]$, so equiv. utility is also α_H.

- For a complex lottery L in which we select either L^+ or L^- with probability 0.5, the equiv. utility is still α_H.

- On the other hand, in L, we get A_1 with probability 0.5 and A_0 with probability 0.5.

- Thus, $L = L(0.5)$ and hence, $u(L) = 0.5$.

- So, we conclude that $\alpha_H = 0.5$.

20.   Which Action Should We Choose?

- Suppose that an action has n possible outcomes $S_1, \ldots, S_n$, with utilities $[\underline{u}(S_i), \overline{u}(S_i)]$, and probabilities $[\underline{p}_i, \overline{p}_i]$.

- We know that each alternative is equivalent to a simple lottery with utility $u_i = \alpha_H \cdot \overline{u}(S_i) + (1 - \alpha_H) \cdot \underline{u}(S_i)$.

- We know that for each i, the i-th event is equivalent to $p_i = \alpha_H \cdot \overline{p}_i + (1 - \alpha_H) \cdot \underline{p}_i$.

- Thus, this action is equivalent to a situation in which we get utility u_i with probability p_i.

- The utility of such a situation is equal to $\sum\limits_{i=1}^{n} p_i \cdot u_i$.

- Thus, the equivalent utility of the original action is equivalent to

$$\sum_{i=1}^{n} \left(\alpha_H \cdot \overline{p}_i + (1 - \alpha_H) \cdot \underline{p}_i \right) \cdot (\alpha_H \cdot \overline{u}(S_i) + (1 - \alpha_H) \cdot \underline{u}(S_i)) \, .$$

21. Observation: the Resulting Decision Depends on the Level of Detail

- Let us consider a situation in which, with some prob. p, we gain a utility u, else we get 0.

- The expected utility is $p \cdot u + (1 - p) \cdot 0 = p \cdot u$.

- Suppose that we only know the intervals $[\underline{u}, \overline{u}]$ and $[\underline{p}, \overline{p}]$.

- The equivalent utility u_k (k for $k$now) is

$$u_k = (\alpha_H \cdot \overline{p} + (1 - \alpha_H) \cdot \underline{p}) \cdot (\alpha_H \cdot \overline{u} + (1 - \alpha_H) \cdot \underline{u}).$$

- If we only know that utility is from $[\underline{p} \cdot \underline{u}, \overline{p} \cdot \overline{u}]$, then:

$$u_d = \alpha_H \cdot \overline{p} \cdot \overline{u} + (1 - \alpha_H) \cdot \underline{p} \cdot \underline{u} \ (d \text{ for } d\text{on't know}).$$

- Here, additional knowledge decreases utility:

$$u_d - u_k = \alpha_H \cdot (1 - \alpha_H) \cdot (\overline{p} - \underline{p}) \cdot (\overline{u} - \underline{u}) > 0.$$

- (This is maybe what the Book of Ecclesiastes meant by "For with much wisdom comes much sorrow"?)

The Notion of Utility

How to Estimate . . .

How to Estimate . . .

From Utility to . . .

From Interval-Valued . . .

Decisions under . . .

Nash's Bargaining . . .

Group Decision . . .

Paradox of Love

The Notion of Utility

How to Estimate . . .

How to Estimate . . .

From Utility to . . .

From Interval-Valued . . .

Decisions under . . .

Nash's Bargaining . . .

Group Decision . . .

Paradox of Love

22. Beyond Interval Uncertainty: Partial Info about Probabilities

- *Frequent situation:*

  - in addition to $\mathbf{x}_i$,

  - we may also have *partial* information about the probabilities of different values $x_i \in \mathbf{x}_i$.

- An *exact* probability distribution can be described, e.g., by its cumulative distribution function

$$F_i(z) = \mathrm{Prob}(x_i \leq z).$$

- A *partial* information means that instead of a single cdf, we have a *class* $\mathcal{F}$ of possible cdfs.

- *p-box* (Scott Ferson):

  - for every z, we know an interval $\mathbf{F}(z) = [\underline{F}(z), \overline{F}(z)]$;

  - we consider all possible distributions for which, for all z, we have $F(z) \in \mathbf{F}(z)$.

23. Describing Partial Info about Probabilities: Decision Making Viewpoint

- *Problem:* there are many ways to represent a probability distribution.

- *Idea:* look for an objective.

- *Objective:* make decisions $E_x[u(x, a)] \to \max\limits_a$.

- *Case 1:* smooth $u(x)$.

- *Analysis:* we have $u(x) = u(x_0) + (x - x_0) \cdot u'(x_0) + \ldots$

- *Conclusion:* we must know moments to estimate $E[u]$.

- *Case of uncertainty:* interval bounds on moments.

- *Case 2:* threshold-type $u(x)$ (e.g., regulations).

- *Conclusion:* we need cdf $F(x) = \text{Prob}(\xi \leq x)$.

- *Case of uncertainty:* p-box $[\underline{F}(x), \overline{F}(x)]$.

24.   What if Intervals are Difficult to Elicit

- *Problem:* in some situations, it is difficult to elicit even interval-valued utilities.

- *Case study:* selecting a location for a meteorological tower.

- *What we can use for decision making:* in many such situations, there are reasonable symmetries.

- *Good news:* in such cases, we can often use symmetries to select an optimal decision.

- *We show:* how this works on the case study example.

The Notion of Utility

How to Estimate. . .

How to Estimate. . .

From Utility to. . .

From Interval-Valued. . .

Decisions under. . .

Nash's Bargaining. . .

Group Decision. . .

Paradox of Love

25.   Case Study

- *Objective:* select the best location of a sophisticated multi-sensor meteorological tower.

- *Constraints:* we have several criteria to satisfy.

- *Example:* the station should not be located too close to a road.

- *Motivation:* the gas flux generated by the cars do not influence our measurements of atmospheric fluxes.

- *Formalization:* the distance x_1 to the road should be larger than a threshold t_1: $x_1 > t_1$, or $y_1 \stackrel{\text{def}}{=} x_1 - t_1 > 0$.

- *Example:* the inclination x_2 at the tower's location should be smaller than a threshold t_2: $x_2 < t_2$.

- *Motivation:* otherwise, the flux determined by this inclination and not by atmospheric processes.

Home Page

Title Page

◀◀   ▶▶

◀   ▶

Page 26 of 62

Go Back

Full Screen

Close

Quit

The Notion of Utility

How to Estimate . . .

How to Estimate . . .

From Utility to . . .

From Interval-Valued . . .

Decisions under . . .

Nash's Bargaining . . .

Group Decision . . .

Paradox of Love

26. General Case

- *In general:* we have several differences $y_1, \ldots, y_n$ all of which have to be non-negative.

- For each of the differences y_i, the larger its value, the better.

- Our problem is a typical setting for *multi-criteria optimization*.

- A most widely used approach to multi-criteria optimization is *weighted average*, where

  - we assign weights $w_1, \ldots, w_n > 0$ to different criteria y_i and

  - select an alternative for which the weighted average

    $$w_1 \cdot y_1 + \ldots + w_n \cdot y_n$$

    attains the largest possible value.

The Notion of Utility

How to Estimate. . .

How to Estimate. . .

From Utility to. . .

From Interval-Valued . . .

Decisions under . . .

Nash's Bargaining . . .

Group Decision . . .

Paradox of Love

27. Towards a Precise Description

- Each alternative is characterized by a tuple of n positive values $y = (y_1, \ldots, y_n)$.

- Thus, the set of all alternatives is the set $(R^+)^n$ of all the tuples of positive numbers.

- For each two alternatives y and y', we want to tell whether

  - y is better than y' (we will denote it by $y \succ y'$ or $y' \prec y$),

  - or y' is better than y ($y' \succ y$),

  - or y and y' are equally good ($y' \sim y$).

- *Natural requirement:* if y is better than y' and y' is better than y'', then y is better than y''.

- The relation $\succ$ must be transitive.

The Notion of Utility

How to Estimate...

How to Estimate...

From Utility to...

From Interval-Valued...

Decisions under...

Nash's Bargaining...

Group Decision...

Paradox of Love

28.   Towards a Precise Description (cont-d)

- *Reminder:* the relation $\succ$ must be transitive.

- Similarly, the relation $\sim$ must be transitive, symmetric, and reflexive ($y \sim y$), i.e., be an *equivalence relation.*

- *An alternative description:* a transitive *pre-ordering* relation $a \succeq b \Leftrightarrow (a \succ b \vee a \sim b)$ s.t. $a \succeq b \vee b \succeq a$.

- Then, $a \sim b \Leftrightarrow (a \succeq b) \,\&\, (b \succeq a)$, and

$$a \succ b \Leftrightarrow (a \succeq b) \,\&\, (b \not\succeq a).$$

- *Additional requirement:*

  - *if* each criterion is better,

  - *then* the alternative is better as well.

- *Formalization:* if $y_i > y'_i$ for all i, then $y \succ y'$.

29.   Scale Invariance: Motivation

- *Fact:* quantities y_i describe completely different physical notions, measured in completely different units.

- *Examples:* wind velocities measured in m/s, km/h, mi/h; elevations in m, km, ft.

- Each of these quantities can be described in many different units.

- A priori, we do not know which units match each other.

- Units used for measuring different quantities may not be exactly matched.

- It is reasonable to require that:
  - if we simply change the units in which we measure each of the corresponding n quantities,
  - the relations $\succ$ and $\sim$ between the alternatives $y = (y_1, \ldots, y_n)$ and $y' = (y'_1, \ldots, y'_n)$ do not change.

30. Scale Invariance: Towards a Precise Description

- *Situation:* we replace:
  - a unit in which we measure a certain quantity q
  - by a new measuring unit which is $\lambda > 0$ times smaller.

- *Result:* the numerical values of this quantity increase by a factor of λ: $q \to \lambda \cdot q$.

- *Example:* 1 cm is $\lambda = 100$ times smaller than 1 m, so the length $q = 2$ becomes $\lambda \cdot q = 2 \cdot 100 = 200$ cm.

- Then, scale-invariance means that for all $y, y' \in (R^+)^n$ and for all $\lambda_i > 0$, we have

  - $y = (y_1, \ldots, y_n) \succ y' = (y'_1, \ldots, y'_n)$ implies $(\lambda_1 \cdot y_1, \ldots, \lambda_n \cdot y_n) \succ (\lambda_1 \cdot y'_1, \ldots, \lambda_n \cdot y'_n)$,
  - $y = (y_1, \ldots, y_n) \sim y' = (y'_1, \ldots, y'_n)$ implies $(\lambda_1 \cdot y_1, \ldots, \lambda_n \cdot y_n) \sim (\lambda_1 \cdot y'_1, \ldots, \lambda_n \cdot y'_n)$.

The Notion of Utility

How to Estimate...

How to Estimate...

From Utility to...

From Interval-Valued...

Decisions under...

Nash's Bargaining...

Group Decision...

Paradox of Love

31.   Formal Description

- By a *total pre-ordering relation* on a set Y, we mean

  - a pair of a transitive relation $\succ$ and an equivalence relation $\sim$ for which,

  - for every $y, y' \in Y$, exactly one of the following relations hold: $y \succ y'$, $y' \succ y$, or $y \sim y'$.

- We say that a total pre-ordering is *non-trivial* if there exist y and y' for which $y \succ y'$.

- We say that a total pre-ordering relation on $(R^+)^n$ is:

  - *monotonic* if $y_i' > y_i$ for all i implies $y' \succ y$;

  - *continuous* if

    * whenever we have a sequence $y^{(k)}$ of tuples for which $y^{(k)} \succeq y'$ for some tuple y', and
    * the sequence $y^{(k)}$ tends to a limit y,
    * then $y \succeq y'$.

32. Main Result

Theorem. *Every non-trivial monotonic scale-inv. continuous total pre-ordering relation on $(R^+)^n$ has the form:*

$$y' = (y_1', \ldots, y_n') \succ y = (y_1, \ldots, y_n) \Leftrightarrow \prod_{i=1}^{n}(y_i')^{\alpha_i} > \prod_{i=1}^{n} y_i^{\alpha_i};$$

$$y' = (y_1', \ldots, y_n') \sim y = (y_1, \ldots, y_n) \Leftrightarrow \prod_{i=1}^{n}(y_i')^{\alpha_i} = \prod_{i=1}^{n} y_i^{\alpha_i},$$

for some constants $\alpha_i > 0$.

Comment: Vice versa,

- for each set of values $\alpha_1 > 0, \ldots, \alpha_n > 0$,

- the above formulas define a monotonic scale-invariant continuous pre-ordering relation on $(R^+)^n$.

The Notion of Utility

How to Estimate...

How to Estimate...

From Utility to...

From Interval-Valued...

Decisions under...

Nash's Bargaining...

Group Decision...

Paradox of Love

33.   Practical Conclusion

- *Situation:*

  - we need to select an alternative;

  - each alternative is characterized by characteristics $y_1, \ldots, y_n$.

- *Traditional approach:*

  - we assign the weights w_i to different characteristics;

  - we select the alternative with the largest value of $\sum_{i=1}^{n} w_i \cdot y_i$.

- *New result:* it is better to select an alternative with the largest value of $\prod_{i=1}^{n} y_i^{w_i}$.

- *Equivalent reformulation:* select an alternative with the largest value of $\sum_{i=1}^{n} w_i \cdot \ln(y_i)$.

Home Page

Title Page

◀◀   ▶▶

◀   ▶

Page 34 of 62

Go Back

Full Screen

Close

Quit

The Notion of Utility

How to Estimate...

How to Estimate...

From Utility to...

From Interval-Valued...

Decisions under...

Nash's Bargaining...

Group Decision...

Paradox of Love

34. Multi-Agent Cooperative Decision Making

- *How to describe preferences:* for each participant P_i, we can determine the utility $u_{ij} \stackrel{\text{def}}{=} u_i(A_j)$ of all A_j.

- *Question:* how to transform these utilities into a reasonable group decision rule?

- *Solution:* was provided by another future Nobelist John Nash.

- *Nash's assumptions:*

    – symmetry,

    – independence from irrelevant alternatives, and

    – *scale invariance* – under replacing function $u_i(A)$ with an equivalent function $a \cdot u_i(A)$,

35. Nash's Bargaining Solution (cont-d)

- *Nash's assumptions (reminder):*

  - symmetry,

  - independence from irrelevant alternatives, and

  - scale invariance.

- *Nash's result:*

  - the only group decision rule satisfying all these assumptions

  - is selecting an alternative A for which the product $\prod_{i=1}^{n} u_i(A)$ is the largest possible.

- *Comment.* the utility functions must be "scaled" s.t. the "status quo" situation $A^{(0)}$ has utility 0:

$$u_i(A) \to u_i'(A) \stackrel{\text{def}}{=} u_i(A) - u_i(A^{(0)}).$$

The Notion of Utility

How to Estimate . . .

How to Estimate . . .

From Utility to . . .

From Interval-Valued . . .

Decisions under . . .

Nash's Bargaining . . .

Group Decision . . .

Paradox of Love

36. Comment: Nash's Solution Can Be Easily Explained in Terms of Fuzzy Logic

- We want all participants to be happy.

- So, we want the first participant to be happy *and* the second participant to be happy, etc.

- We can take:

  - $u_1(A)$ as the "degree of happiness" of the first participant,

  - $u_2(A)$ as the "degree of happiness" of the second participant, etc.

- To formalize "and", we use $d \cdot d'$ (one of the two "and"-operations originally proposed by L. Zadeh).

- Then, the degree to which all n participants are satisfied is equal to the product $u_1(A) \cdot u_2(A) \cdot \ldots \cdot u_n(A)$.

Home Page

Title Page

◀◀ ▶▶

◀ ▶

Page 37 of 62

Go Back

Full Screen

Close

Quit

The Notion of Utility

How to Estimate . . .

How to Estimate . . .

From Utility to . . .

From Interval-Valued . . .

Decisions under . . .

Nash's Bargaining . . .

Group Decision . . .

Paradox of Love

37. Multi-Agent Decision Making under Interval Uncertainty

- *Reminder:* if we set utility of status quo to 0, then we select an alternative A that maximizes

$$u(A) = \prod_{i=1}^{n} u_i(A).$$

- *Case of interval uncertainty:* we only know intervals $[\underline{u}_i(A), \overline{u}_i(A)]$.

- *First idea:* find all A_0 for which $\overline{u}(A_0) \geq \max_A \underline{u}(A)$, where

$$[\underline{u}(A), \overline{u}(A)] \stackrel{\text{def}}{=} \prod_{i=1}^{n} [\underline{u}_i(A), \overline{u}_i(A)].$$

- *Second idea:* maximize $u^{\text{equiv}}(A) \stackrel{\text{def}}{=} \prod_{i=1}^{n} u_i^{\text{equiv}}(A)$.

- *Interesting aspect:* when we have a conflict situation (e.g., in security games).

The Notion of Utility

How to Estimate. . .

How to Estimate. . .

From Utility to. . .

From Interval-Valued. . .

Decisions under. . .

Nash's Bargaining. . .

Group Decision. . .

Paradox of Love

38. Group Decision Making and Arrow's Impossibility Theorem

- In 1951, Kenneth J. Arrow published his famous result about group decision making.

- This result that became one of the main reasons for his 1972 Nobel Prize.

- *The problem:*

  - A group of n participants $P_1, \ldots, P_n$ needs to select between one of m alternatives $A_1, \ldots, A_m$.

  - To find individual preferences, we ask each participant P_i to rank the alternatives A_j:

  $$A_{j_1} \succ_i A_{j_2} \succ_i \ldots \succ_i A_{j_n}.$$

  - Based on these n rankings, we must form a single group ranking (equivalence $\sim$ is allowed).

The Notion of Utility

How to Estimate . . .

How to Estimate . . .

From Utility to . . .

From Interval-Valued . . .

Decisions under . . .

Nash's Bargaining . . .

Group Decision . . .

Paradox of Love

39. Case of Two Alternatives Is Easy

- *Simplest case:*

  - we have only two alternatives A_1 and A_2,

  - each participant either prefers A_1 or prefers A_2.

- *Solution:* it is reasonable, for a group:

  - to prefer A_1 if the majority prefers A_1,

  - to prefer A_2 if the majority prefers A_2, and

  - to claim A_1 and A_2 to be of equal quality for the group (denoted $A_1 \sim A_2$) if there is a tie.

Home Page

Title Page

◀◀   ▶▶

◀   ▶

Page 40 of 62

Go Back

Full Screen

Close

Quit

The Notion of Utility

How to Estimate. . .

How to Estimate. . .

From Utility to. . .

From Interval-Valued . . .

Decisions under . . .

Nash's Bargaining . . .

Group Decision . . .

Paradox of Love

40. Case of Three or More Alternatives Is Not Easy

- *Arrow's result:* no group decision rule can satisfy the following natural conditions.

- *Pareto condition:* if all participants prefer A_j to A_k, then the group should also prefer A_j to A_k.

- *Independence from Irrelevant Alternatives:* the group ranking of A_j vs. A_k should not depend on other A_is.

- *Arrow's theorem:* every group decision rule which satisfies these two condition is a *dictatorship* rule:
  - the group accepts the preferences of one of the participants as the group decision and
  - ignores the preferences of all other participants.

- This violates *symmetry*: that the group decision rules should not depend on the order of the participants.

The Notion of Utility

How to Estimate . . .

How to Estimate . . .

From Utility to . . .

From Interval-Valued . . .

Decisions under . . .

Nash's Bargaining . . .

Group Decision . . .

Paradox of Love

41. Beyond Arrow's Impossibility Theorem

- *Usual claim:* Arrow's Impossibility Theorem proves that reasonable group decision making is impossible.

- *Our claim:* Arrow's result is only valid if we have binary ("yes"-"no") individual preferences.

- *Fact:* this information does not fully describe a persons' preferences.

- *Example:* the preference $A_1 \succ A_2 \succ A_3$:

  - it may indicate that a person strongly prefers A_1 to A_2, and strongly prefers A_2 to A_3, and

  - it may also indicate that this person strongly prefers A_1 to A_2, and at the same time, $A_2 \approx A_3$.

- *How can this distinction be described:* researchers in decision making use the notion of *utility*.

The Notion of Utility

How to Estimate...

How to Estimate...

From Utility to...

From Interval-Valued...

Decisions under...

Nash's Bargaining...

Group Decision...

Paradox of Love

42. Nash's Solution as a Way to Overcome Arrow's Paradox

- *Situation:* for each participant P_i $(i = 1, \ldots, n)$, we know his/her utility $u_i(A_j)$ of A_j, $j = 1, \ldots, m$.

- *Possible choices:* lotteries $p = (p_1, \ldots, p_m)$ in which we select A_j with probability $p_j \geq 0$, $\sum\limits_{j=1}^{m} p_j = 1$.

- *Nash's solution:* among all the lotteries p, we select the one that maximizes

$$\prod_{i=1}^{n} u_i(p), \text{ where } u_i(p) = \sum_{j=1}^{m} p_j \cdot u_i(A_j).$$

- *Generic case:* no two vectors $u_i = (u_i(A_1), \ldots, u_i(A_m))$ are collinear.

- *In this general case:* Nash's solution is unique.

Home Page

Title Page

◀◀ ▶▶

◀ ▶

Page 43 of 62

Go Back

Full Screen

Close

Quit

The Notion of Utility

How to Estimate . . .

How to Estimate . . .

From Utility to . . .

From Interval-Valued . . .

Decisions under . . .

Nash's Bargaining . . .

Group Decision . . .

Paradox of Love

43.   Sometimes It Is Beneficial to Cheat: An Example

- *Situation:* participant P_1 know the utilities of all the other participants, but they don't know his $u_1(B)$.

- *Notation:* let A_{m_1} be P_1's best alternative:

$$u_1(A_{m_1}) \geq u_1(A_j) \text{ for all } j \neq m_1.$$

- *How to cheat:* P_1 can force the group to select A_{m_1} by using a "fake" utility function $u_1'(A)$ for which

  - $u_1'(A_{m_1}) = 1$ and
  - $u_1'(A_j) = 0$ for all $j \neq m_1$.

- *Why it works:*

  - when selecting A_j w/$j \neq m_1$, we get $\prod u_i(A_j) = 0$;
  - when selecting A_{m_1}, we get $\prod u_i(A_j) > 0$.

The Notion of Utility

How to Estimate . . .

How to Estimate . . .

From Utility to . . .

From Interval-Valued . . .

Decisions under . . .

Nash's Bargaining . . .

Group Decision . . .

Paradox of Love

Home Page

Title Page

◀◀ ▶▶

◀ ▶

Page 45 of 62

Go Back

Full Screen

Close

Quit

44. Cheating May Hurt the Cheater: an Observation

- *A more typical situation:* no one knows others' utility functions.

- Let P_1 use the above false utility function $u'_1(A)$ for which $u'_1(A_{m_1}) = 1$ and $u'_1(A_j) = 0$ for all $j \neq m_1$.

- *Possibility:* others use similar utilities with $u_i(A_{m_i}) > 0$ for some $m_i \neq m_1$ and $u_i(A_j) = 0$ for $j \neq m_i$.

- Then for every alternative A_j, Nash's product is equal to 0.

- From this viewpoint, all alternatives are equally good, so each of them can be chosen.

- In particular, it may be possible that the group selects an alternative A_q which is *the worst* for P_1 – i.e., s.t.

$$u_1(A_q) < u_1(A_j) \text{ for all } j \neq p.$$

The Notion of Utility

How to Estimate. . .

How to Estimate. . .

From Utility to. . .

From Interval-Valued. . .

Decisions under. . .

Nash's Bargaining. . .

Group Decision . . .

Paradox of Love

45. Case Study: Territorial Division

- Dividing a set (territory) A between n participants, i.e., finding X_i s.t. $\bigcup\limits_{i=1}^{n} X_i$ and $X_i \cap X_j = \emptyset$ for $i \neq j$.

- The utility functions have the form $u_i(X) = \int_X v_i(t)\, dt$.

- *Nash's solution:* maximize $u_1(X) \cdot \ldots \cdot u_n(X_n)$.

- *Assumption:* P_1 does not know $u_i(B)$ for $i \neq 1$.

- *Choices:* the participant P_1 can report a fake utility function $v_1'(t) \neq v_1(t)$.

- For each $v_1'(t)$, we maximizes the product

$$\left(\int_{X_1} v_1'(t)\, dt \right) \cdot \left(\int_{X_2} v_2(t)\, dt \right) \cdot \ldots \cdot \left(\int_{X_n} v_n(t)\, dt \right).$$

- *Question:* select $v_1'(t)$ that maximizes the gain

$$u(v_1', v_1, v_2, \ldots, v_n) \stackrel{\text{def}}{=} \int_{X_1} v_1'(t)\, dt.$$

46. For Territorial Division, It Is Beneficial to Report the Correct Utilities: Result

- *Hurwicz's criterion* $u(A) = \alpha \cdot u^-(A) + (1 - \alpha) \cdot u^+(A)$ may sound arbitrary.

- *For our problem:* Hurwicz's criterion means that we select a utility function $v_1'(t)$ that maximizes

$$J(v_1') \stackrel{\text{def}}{=} \alpha \cdot \min_{v_2,\ldots,v_n} u(v_1', v_1, v_2, \ldots, v_n) +$$

$$(1 - \alpha) \cdot \max_{v_2,\ldots,v_n} u(v_1', v_1, v_2, \ldots, v_n).$$

- *Theorem:* when $\alpha > 0$, the objective function $J(v_1')$ attains its largest possible value for $v_1'(t) = v_1(t)$.

- *Conclusion:* unless we select pure optimism, it is best to select $v_1'(t) = v_1(t)$, i.e., to tell the truth.

Home Page

Title Page

◀◀ ▶▶

◀ ▶

Page 48 of 62

Go Back

Full Screen

Close

Quit

47. How to Find Individual Preferences from Collective Decision Making: Inverse Problem of Game Theory

- *Situation:* we have a group of n participants $P_1, \ldots, P_n$ that does not want to reveal its individual preferences.

- *Example:* political groups tend to hide internal disagreements.

- *Objective:* detect individual preferences.

- *Example:* this is waht kremlinologies used to do.

- *Assumption:* the group uses Nash's solution to make decisions.

- *We can:* ask the group as a whole to compare different alternatives.

The Notion of Utility

How to Estimate. . .

How to Estimate. . .

From Utility to. . .

From Interval-Valued. . .

Decisions under . . .

Nash's Bargaining . . .

Group Decision . . .

Paradox of Love

48. Comment

- *Fact:* Nash's solution depends only on the product of the utility functions.

- *Corollary:* in the best case,

  - we will be able to determine n individual utility functions

  - without knowing which of these functions corresponds to which individual.

- *Comment:* this is OK, because

  - our main objective is to predict future behavior of this group,

  - and in this prediction, it is irrelevant who has which utility function.

Home Page

Title Page

◀◀   ▶▶

◀   ▶

Page 49 of 62

Go Back

Full Screen

Close

Quit

49. How to Find Individual Preferences from Collective Decision Making: Our Result

- Let $u_{ij} = u_i(A_j)$ denote i-th utility of j-th alternative.

- *We assume* that utility is normalized: $u_i(A_0) = 0$ for status quo A_0 and $u_i(A_1) = 1$ for some A_1.

- *According to Nash:* $p = (p_1, \ldots, p_n) \preceq q = (q_1, \ldots, q_n) \Leftrightarrow$

$$\prod_{i=1}^{n} \left(\sum_{j=1}^{n} p_j \cdot u_{ij} \right) \leq \prod_{i=1}^{n} \left(\sum_{j=1}^{n} q_j \cdot u_{ij} \right).$$

- *Theorem:* if utilities u_{ij} and u'_{ij} lead to the same preference $\preceq$, then they differ only by permutation.

- *Conclusion:* we can determine individual preferences from group decisions.

- *An efficient algorithm* for determining u_{ij} from $\preceq$ is possible.

The Notion of Utility

How to Estimate . . .

How to Estimate . . .

From Utility to . . .

From Interval-Valued . . .

Decisions under . . .

Nash's Bargaining . . .

Group Decision . . .

Paradox of Love

Home Page

Title Page

◀◀   ▶▶

◀   ▶

Page 51 of 62

Go Back

Full Screen

Close

Quit

50.   We Must Take Altruism and Love into Account

- *Implicit assumption:* the utility $u_i(A_j)$ of a participant P_i depends only on what he/she gains.

- *In real life:* the degree of a person's happiness also depends on the degree of happiness of other people:

    - Normally, this dependence is positive, i.e., we feel happier if other people are happy.
    - However, negative emotions such as jealousy are also common.

- This idea was developed by another future Nobelist Gary Becker: $u_i = u_i^{(0)} + \sum_{j \neq i} \alpha_{ij} \cdot u_j$, where:

    - $u_i^{(0)}$ is the utility of person i that does not take interdependence into account; and
    - u_j are utilities of other people $j \neq i$.

51. Paradox of Love

- *Case $n = 2$:* $u_1 = u_1^{(0)} + \alpha_{12} \cdot u_2$; $u_2 = u_2^{(0)} + \alpha_{21} \cdot u_1$.

- *Solution:* $u_1 = \dfrac{u_1^{(0)} + \alpha_{12} \cdot u_2^{(0)}}{1 - \alpha_{12} \cdot \alpha_{21}}$; $u_2 = \dfrac{u_2^{(0)} + \alpha_{21} \cdot u_1^{(0)}}{1 - \alpha_{12} \cdot \alpha_{21}}$.

- *Example:* mutual affection means that $\alpha_{12} > 0$ and $\alpha_{21} > 0$.

- *Example:* selfless love, when someone else's happiness means more than one's own, corresponds to $\alpha_{12} > 1$.

- *Paradox:*

  - when two people are deeply in love with each other ($\alpha_{12} > 1$ and $\alpha_{21} > 1$), then

  - positive original pleasures $u_i^{(0)} > 0$ lead to $u_i < 0$ – i.e., to unhappiness.

52. Paradox of Love: Discussion

- *Paradox – reminder:*

  - when two people are deeply in love with each other, then

  - positive original pleasures $u_i^{(0)} > 0$ lead to unhappiness.

- This may explain why people in love often experience deep negative emotions.

- From this viewpoint, a situation when

  - one person loves deeply and

  - another rather allows him- or herself to be loved

  may lead to more happiness than mutual passionate love.

The Notion of Utility

How to Estimate...

How to Estimate...

From Utility to...

From Interval-Valued...

Decisions under...

Nash's Bargaining...

Group Decision...

Paradox of Love

53.   Why Two and not Three?

- An *ideal love* is when each person treats other's emotions almost the same way as one's own, i.e.,

$$\alpha_{12} = \alpha_{21} = \alpha = 1 - \varepsilon \text{ for a small } \varepsilon > 0.$$

- For *two people*, from $u_i^{(0)} > 0$, we get $u_i > 0$ – i.e., we can still have happiness.

- For $n \geq 3$, even for $u_i^{(0)} = u^{(0)} > 0$, we get

$$u_i = \frac{u^{(0)}}{1 - (1 - \varepsilon) \cdot (n - 1)} < 0, \text{ i.e., unhappiness.}$$

- *Corollary:* if two people care about the same person (e.g., his mother and his wife),

  - all three of them are happier

  - if there is some negative feeling (e.g., jealousy) between them.

54.   Emotional vs. Objective Interdependence

- *We considered:* emotional interdependence, when one's utility is determined by the utility of other people:

$$u_i = u_i^{(0)} + \sum_j \alpha_j \cdot u_j.$$

- *Alternative:* "objective" altruism, when one's utility depends on the material gain of other people:

$$u_i = u_i^{(0)} + \sum_j \alpha_j \cdot u_j^{(0)}.$$

- *In this approach:* we care about others' well-being, not about their emotions.

- *In this approach:* no paradoxes arise, any degree of altruism only improves the situation.

- The objective approach was proposed by yet another Nobel Prize winner Amartya K. Sen.

55. Acknowledgments

This work was supported in part:

The Notion of Utility

How to Estimate...

How to Estimate...

From Utility to...

From Interval-Valued...

Decisions under...

Nash's Bargaining...

Group Decision...

Paradox of Love

56. Proof of Symmetry Result: Part 1

- Due to scale-invariance, for every $y_1, \ldots, y_n, y'_1, \ldots, y'_n$, we can take $\lambda_i = \dfrac{1}{y_i}$ and conclude that

$$(y'_1, \ldots, y'_n) \sim (y_1, \ldots, y_n) \Leftrightarrow \left(\frac{y'_1}{y_1}, \ldots, \frac{y'_n}{y_n}\right) \sim (1, \ldots, 1).$$

- Thus, to describe the equivalence relation $\sim$, it is sufficient to describe $\{z = (z_1, \ldots, z_n) : z \sim (1, \ldots, 1)\}$.

- Similarly,

$$(y'_1, \ldots, y'_n) \succ (y_1, \ldots, y_n) \Leftrightarrow \left(\frac{y'_1}{y_1}, \ldots, \frac{y'_n}{y_n}\right) \succ (1, \ldots, 1).$$

- Thus, to describe the ordering relation $\succ$, it is sufficient to describe the set $\{z = (z_1, \ldots, z_n) : z \succ (1, \ldots, 1)\}$.

- Similarly, it is also sufficient to describe the set

$$\{z = (z_1, \ldots, z_n) : (1, \ldots, 1) \succ z\}.$$

57.   Proof of Symmetry Result: Part 2

- *To simplify:* take logarithms $Y_i = \ln(y_i)$, and sets

$$S_\sim = \{Z : z = (\exp(Z_1), \ldots, \exp(Z_n)) \sim (1, \ldots, 1)\},$$

$$S_\succ = \{Z : z = (\exp(Z_1), \ldots, \exp(Z_n)) \succ (1, \ldots, 1)\};$$

$$S_\prec = \{Z : (1, \ldots, 1) \succ z = (\exp(Z_1), \ldots, \exp(Z_n))\}.$$

- Since the pre-ordering relation is total, for Z, either $Z \in S_\sim$ or $Z \in S_\succ$ or $Z \in S_\prec$.

- *Lemma:* $S_\sim$ is closed under addition:

  - $Z \in S_\sim$ means $(\exp(Z_1), \ldots, \exp(Z_n)) \sim (1, \ldots, 1)$;

  - due to scale-invariance, we have

  $$(\exp(Z_1+Z_1'), \ldots) = (\exp(Z_1) \cdot \exp(Z_1'), \ldots) \sim (\exp(Z_1'), \ldots);$$

  - also, $Z' \in S_\sim$ means $(\exp(Z_1'), \ldots) \sim (1, \ldots, 1)$;

  - since $\sim$ is transitive,

  $$(\exp(Z_1 + Z_1'), \ldots) \sim (1, \ldots) \text{ so } Z + Z' \in S_\sim.$$

The Notion of Utility

How to Estimate . . .

How to Estimate . . .

From Utility to . . .

From Interval-Valued . . .

Decisions under . . .

Nash's Bargaining . . .

Group Decision . . .

Paradox of Love

Home Page

Title Page

◀◀   ▶▶

◀   ▶

Page 59 of 62

Go Back

Full Screen

Close

Quit

58. Proof of Symmetry Result: Part 3

- *Reminder:* the set $S_\sim$ is closed under addition;

- Similarly, $S_\prec$ and $S_\succ$ are closed under addition.

- *Conclusion:* for every integer $q > 0$:

  − if $Z \in S_\sim$, then $q \cdot Z \in S_\sim$;

  − if $Z \in S_\succ$, then $q \cdot Z \in S_\succ$;

  − if $Z \in S_\prec$, then $q \cdot Z \in S_\prec$.

- Thus, if $Z \in S_\sim$ and $q \in N$, then $(1/q) \cdot Z \in S_\sim$.

- We can also prove that $S_\sim$ is closed under $Z \to -Z$:

  - $Z = (Z_1, \ldots) \in S_\sim$ means $(\exp(Z_1), \ldots) \sim (1, \ldots)$;

  - by scale invariance, $(1, \ldots) \sim (\exp(-Z_1), \ldots)$, i.e., $-Z \in S_\sim$.

- Similarly, $Z \in S_\succ \Leftrightarrow -Z \in S_\prec$.

- So $Z \in S_\sim \Rightarrow (p/q) \cdot Z \in S_\sim$; in the limit, $x \cdot Z \in S_\sim$.

59.   Proof of Symmetry Result: Final Part

- *Reminder:* $S_\sim$ is closed under addition and multiplication by a scalar, so it is a linear space.

- *Fact:* $S_\sim$ cannot have full dimension n, since then all alternatives will be equivalent to each other.

- *Fact:* $S_\sim$ cannot have dimension $< n - 1$, since then:

  - we can select an arbitrary $Z \in S_\prec$;
  - connect it w/$-Z \in S_\succ$ by a path γ that avoids $S_\sim$;
  - due to closeness, $\exists \gamma(t^*)$ in the limit of $S_\succ$ and $S_\prec$;
  - thus, $\gamma(t^*) \in S_\sim$ – a contradiction.

- Every $(n-1)$-dim lin. space has the form $\sum\limits_{i=1}^{n} \alpha_i \cdot Y_i = 0$.

- Thus, $Y \in S_\succ \Leftrightarrow \sum \alpha_i \cdot Y_i > 0$, and

$$y \succ y' \Leftrightarrow \sum \alpha_i \cdot \ln(y_i/y_i') > 0 \Leftrightarrow \prod y_i^{\alpha_i} > \prod y_i'^{\alpha_i}.$$

The Notion of Utility

How to Estimate...

How to Estimate...

From Utility to...

From Interval-Valued...

Decisions under...

Nash's Bargaining...

Group Decision...

Paradox of Love

Home Page

Title Page

◀◀ ▶▶

◀ ▶

Go Back

Full Screen

Close

Quit

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