# Introduction to Fuzzy Logic

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# 1. Need for Expert Knowledge

- In some cases, we have a precise knowledge – e.g., an autopilot perfectly pilots a plane.
- In many other cases, we have to rely on expert knowledge.
- So far, computer-based systems have not (yet) replaced skilled medical doctors or even skilled drivers.
- In the ideal world, everyone should go to the best doctor.
- However, in real life, this is not possible.
- It is therefore desirable to have a computer-based tool that contains the knowledge of the best doctors.
- Such tool will help all other doctors make good decisions.

## 2. Need to Describe Imprecise (Fuzzy) Knowledge

- Many doctors (and experts in general) are absolutely willing to share their knowledge.
- Challenge: this knowledge is often described in terms of imprecise ("fuzzy") words from natural language.
- A medical doctor can say "if a patient has high fever" or "if a skin mole has an irregular shape".
- A driver cannot say with what force to hit the brakes if the car 10 m in front slows down from 100 to 90 km/h.
- He/she will say "brake a little bit".
- We thus need to translate these fuzzy words into computer-understandable language. about the object we can use.
- This is what Zadeh's *fuzzy logic* is about.


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## 3. Toy Example: a Thermostat

- To illustrate the main idea of fuzzy logic, let us consider a simplified thermostat with a dial.
- Turning the dial to the *left* makes it *cooler*.
- Turning it to the *right* makes it *warmer*.
- We want to reach a comfort temperature $T_0$.
- In other words, we want the difference $x = T - T_0$ to be 0.
- We need to describe, for each $x$, to which angle $u$ we turn the dial: $u = f(x)$.

# 4. Thermostat: Rules

- For such an easy system, we do not need any expert to formulate reasonable rules.
- We can immediately describe several reasonable control rules.
- If the room is comfortable, no control is needed.
- So, if the difference $x = T - T_0$ is negligible, then the control $u$ should also be negligible.
- If the room is slightly overheated, cool it a little bit.
- So, if $x$ is positive and small, $u$ must be negative and small.
- If the temperature is a little lower than we would like it to be, then we need to heat the room a little bit.
- In other terms, if $x$ is small negative, then $u$ must be small positive.

## 5. Thermostat: Rules (cont-d)

- We can formulate many similar natural rules.
- For simplicity, we will restrict ourselves to the above three:
  - if $x$ is negligible, then $u$ must be negligible;
  - if $x$ is small positive, then $u$ must be small negative;
  - if $x$ is small negative, then $u$ must be small positive.

## 6. Rules: General Case

- Let us denote "negligible" by $N$, "small positive" by SP, and "small negative" by SN.
- Then, the rules take the following form:

$$N(x) \Rightarrow N(u); \quad SP(x) \Rightarrow SN(u); \quad SN(x) \Rightarrow SP(u).$$

- In general, the expert's knowledge about the dependence of $y$ on $x_1, \ldots, x_n$ can be expressed by rules:

$$\text{If } x_1 \text{ is } A_{r1}, \ldots, \text{ and } x_n \text{ is } A_{rn}, \text{ then } y \text{ is } B_r.$$

- Here, $A_{ri}$ and $B_r$ are words from natural language like "small", "medium", "large", "approximately 1".
- These rules have the form

$$A_{r1}(x_1)\,\&\,\ldots\,\&\,A_{rn}(x_n) \Rightarrow B_r(y).$$

# 7. Step-by-Step Translation of These Rules

- Our goal is to represent rule bases in precise terms.
- A rule base has a clear structure.
- A rule base consists of rules.
- Each rule, in its turn, is obtained:
  - from properties (expressed by words from natural language)
  - by using logical connectives.
- In view of this structure, it is reasonable to represent the rule base:
  - by first representing the basic elements of the rule base, and then
  - by extending this representation to the rule base as a whole.

## 8. Step-by-Step Translation (cont-d)

- So, first, we represent the properties $A_{ri}(x_i)$ and $B_r(y)$.
- Second, we represent the logical connectives.
- Third, we use logical connectives to represent each rule.
- Fourth, we combine the representations of different rules into a representation of a rule base.
- As a result of these four steps, we get an advising (expert) system.
- For example, if we apply these four steps to the medical knowledge, we ideally, get a system that
  - given the patient's symptoms,
  - provides the diagnostic and medical advice.

# 9. Step-by-Step Translation (cont-d)

- For example, it can say that most probably, the patient has a flu, but it is also possible that he has bronchitis.
- Such an advice, coming from an expert system, is usually used by a specialist to make a decision.
- However, there are situations like automatic control where there is no time to involve a human operator.
- For such control situations, we need an *additional*, fifth follow-up step: making a decision.
- Let us describe all five steps.

## 10. First Step: Representing Natural-language Properties

- For properties $A$ like “small”, for some values $x$, we are not 100% sure whether this value is small or not.
- A natural idea is to ask the expert to mark, on a scale from 0 to 1, to what extend the given value $x$ is small.
- We can use another scale – e.g., 0 to 10 – and then divide by 10.
- As a result, for several values $x_i$, we get a *degree* $A(x_i)$ to which $x_i$ satisfies the property $A$.
- Some experts are not comfortable marking this value.
- Then, we poll the experts and take $A(x_i) = m/n$ if $m$ out of $n$ consider $x_i$ to be, e.g., small.

## 11. Need for Interpolation

- To get the values $A(x)$ for all other $x$, we use interpolation.
- The resulting function is called a *membership function* or a *fuzzy set*.
- The simplest is linear interpolation.
- Let us consider the word "negligible".
- The only case when we are 100% sure that $x$ is negligible is when $x = 0$.
- So, we have $N(0) = 1$.
- Usually, we also know the value $\Delta > 0$ after which the difference in temperatures is no longer negligible.
- For example, for a thermostat that controls the room's temperature, we can take $\Delta = 10F$.

## 12. Interpolation (cont-d)

- This means that $N(x) = 0$ for $x \geq \Delta$ and for $x < -\Delta$.
- We know the value of the function $N(x)$ for $x \leq -\Delta$, for $x = 0$, and for $x \geq \Delta$.
- For $x \in (-\Delta, 0)$, we get the expression

$$N(x) = 1 + \frac{x}{\Delta}.$$

- For $x \in (0, \Delta)$, we get the expression $N(x) = 1 - \dfrac{x}{\Delta}$.
- The graph of $N(x)$ is a triangle.
- Such functions are called *triangular*.

# 13. Need to Combine Fuzzy Degrees

- Conclusions based on expert knowledge often take into account *several* expert statements.
- Our degree of confidence in such a conclusion is thus equal to our degree of confidence that:
  - the first of used statements is true *and*
  - the second used statement is true, etc.
- In other words,
  - in addition to the expert's degrees of confidence in their statements $S_1, \ldots, S_n$,
  - we also need to estimate the degrees of confidence in "and"-combinations $S_i \,\&\, S_j$, $S_i \,\&\, S_j \,\&\, S_k$, etc.
- In the ideal world, we can ask the experts to estimate the degree of confidence in each such combination.
- However, this is not realistically possible.

# 14. Need to Combine (cont-d)

- Problem: for $n$ original statements, there are $2^n - 1$ such combinations.
- Indeed, combinations are in 1-1 correspondence with non-empty subsets of the set of $n$ statements.
- Already for reasonable $n = 30$, we get an astronomical number $2^{30} \approx 10^9$ combinations.
- There is no way that we can ask a billion questions to the experts.
- We cannot elicit the expert's degree of confidence in "and"-combinations directly from the experts.
- So, we need to estimate these degrees based on the experts' degrees of confidence in each statement $S_i$.

## 15. Need to Combine (cont-d)

- In other words, we need to be able:
  - to combine the degrees of confidence $a$ and $b$ of statements $A$ and $B$
  - into an estimate for degree of confidence in the "and"-combination $A \,\&\, B$.
- The algorithm for such combination is called an *"and"-operation* or, for historical reasons, a *t-norm.*
- The result of applying this combination algorithm to numbers $a$ and $b$ will be denoted $f_{\&}(a, b)$.

## 16. How to Combine Fuzzy Degrees?

- Which operation $f_\&(a,b)$ should we choose?
- First, since $A \,\&\, B$ means the same as $B \,\&\, A$, it is reasonable to require that the resulting estimates coincide:

$$f_\&(a,b) = f_\&(b,a).$$

- Since $A \,\&\, (B \,\&\, C)$ means the same as$(A \,\&\, B) \,\&\, C$, we must have

$$f_\&(a, f_\&(b,c)) = f_\&(f_\&(a,b), c).$$

- Since $A \,\&\, B$ is a stronger statement than each of $A$ and $B$ (it implies both $A$ and $B$),

$$f_\&(a,b) \le a \text{ and } f_\&(a,b) \le b.$$

## 17. How to Combine Fuzzy Degrees (cont-d)

- Finally:
  - if our degree of confidence in one or both of the statements $A$ and $B$ increases,
  - the resulting degree of confidence in $A \,\&\, B$ should also increase – or at least remain the same:

$$\text{if } a \le a' \text{ and } b \le b', \text{ then } f_{\&}(a, b) \le f_{\&}(a', b').$$

- There are many such operations: $\min(a, b)$, $a \cdot b$, etc.
- We need to select the one that best represents human reasoning in a given knowledge domain.
- We can also require that since $A \,\&\, A$ means the same as $A$, it is reasonable to require that $f_{\&}(a, a) = a$.
- Then, the only possible t-norm is $\min(a, b)$.
- Similarly, we can define "or"-operations (aka t-conorms).

## 18. Historical Comment

- Historically the first determination of "and"-operation was done at Stanford.
- Researchers designed an expert system MYCIN for diagnosing rare blood diseases.
- The results were good, so they thought that they have discovered general laws of human reasoning.
- However, when they applied the same laws to geophysics, they failed.

## 19. Historical Comment (cont-d)

- Indeed, reasoning is different in these two areas.
- In medicine, we need to be absolutely sure before we recommend, e.g., surgery – else we hurt the patient.
- In geophysics, if there is a good chance to find oil, we start digging.
- If we wait until we are absolutely sure, competitors will be there first.
- So, now we know that in different domains, different t-norms are needed.

## 20. Back to Fuzzy Rules and Toy Example

- Let us go back to our rules

$$N(x) \Rightarrow N(u); \quad SP(x) \Rightarrow SN(u); \quad SN(x) \Rightarrow SP(u).$$

- For each input $x$, the control $u$ is reasonable $R(x, u)$ if one of the rules if applicable:
  - either the 1st rule is applicable, so $N(x)$ and $N(u)$,
  - or the 2nd rule is applicable, so $SP(x)$ and $SN(u)$,
  - or the 3rd rule is applicable, so $SN(x)$ and $SP(u)$:

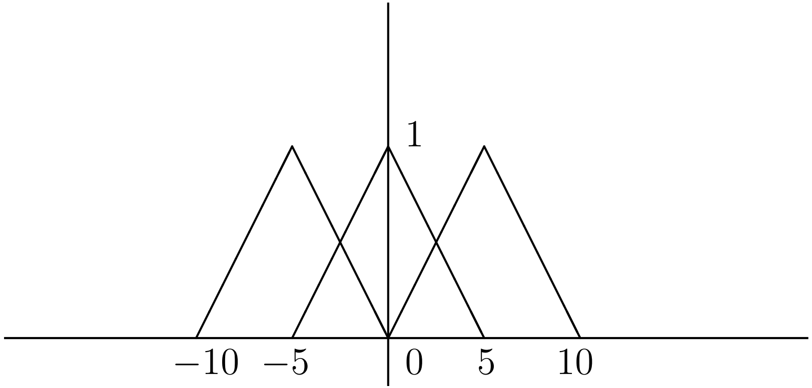
$$R(x,u) \Leftrightarrow (N(x) \,\&\, N(u)) \vee (SP(x) \,\&\, SN(u)) \vee (SN(x) \,\&\, SP(u)).$$

- From experts, we get the values $N(x)$, $N(u)$, etc.
- Based on experts, we select "and"- and "or"-operations.
- Thus, for each $x$ and $u$, we get $R(x, u)$ as

$$f_\vee(f_\&(N(x), N(u)), f_\&(SP(x), SN(u)), f_\&(SN(x), SP(u))).$$

## 21. Numerical Example

- Let us assume that all three membership functions are piece-wise linear.
- Specifically, let us assume that they are described by the following graph:



## 22. Numerical Example (cont-d)

- What is the degree of confidence $\mu_C(4, -2)$ that for $x = 4^\circ$ the control $u = -2^\circ$ is reasonable?
- According to our formulas, let us first compute the values of the membership functions.
- By linear interpolation, we can find the analytical formulas for these membership functions:
- The term "negligible" is described by the following formulas:
  - $\mu_N(x) = 1 + x/5$ for $-5 \le x \le 0$;
  - $\mu_N(x) = 1 - x/5$ for $0 \le x \le 5$;
  - $\mu_N(x) = 0$ for all other $x$.

# 23. Numerical Example (cont-d)

- The term "small positive" is described by the following formulas:
  - $\mu_{SP}(x) = x/5$ for $0 \le x \le 5$;
  - $\mu_{SP}(x) = 2 - x/5$ for $5 \le x \le 10$;
  - $\mu_{SP}(x) = 0$ for all other $x$.
- The term "small negative" is described by the following formulas:
  - $\mu_{SN}(x) = 2 + x/5$ for $-10 \le x \le -5$;
  - $\mu_{SN}(x) = -x/5$ for $-5 \le x \le 0$;
  - $\mu_{SN}(x) = 0$ for all other $x$.

## 24. Numerical Example (cont-d)

- If we use $f_\&(a,b) = \min(a,b)$ and $f_\vee(a,b) = \max(a,b)$, then we get $\mu_C(4,-2) = \max(d_1,d_2,d_3)$, where

$$d_1 = \min(\mu_N(4), \mu_N(-2));\ \ d_2 = \min(\mu_{SP}(4), \mu_{SN}(-2));$$

$$d_3 = \min(\mu_{SN}(4), \mu_{SP}(-2)).$$

- Here, $\mu_N(4) = 0.2$, $\mu_N(-2) = 0.6$, $\mu_S P(4) = 0.8$, $\mu_{SN}(-2) = 0.4$, and $\mu_{SN}(4) = \mu_{SP}(-2) = 0$.

- Hence, $d_1 = \min(0.2, 0.6) = 0.2$, $d_2 = \min(0.8, 0.4) = 0.4$, $d_3 = \min(0, 0) = 0$, and

$$\mu_C(4,-2) = \max(0.2, 0.4, 0) = 0.4.$$

## 25. Defuzzification

- For each $x$ and for each $u$, we get a degree $\mu(u) = R(x, u)$ to which $u$ is reasonable.
- In other words, we get a fuzzy set of possible controls.
- For automatic control, we need to "defuzzify" this into a single value $\overline{u}$.
- For each expert's opinion $u$, we wan to have $\overline{u} - u \approx 0$.
- The vector formed by the differences should be as close to 0 as possible, so we minimize $\sum(\overline{u} - u)^2$.

## 26. Defuzzification (cont-d)

- For each $u$, the degree $\mu(u)$ is proportional to the number of experts who believe that $u$ is reasonable.
- So, the minimized sum becomes $\sim \sum\limits_{u} \mu(u) \cdot (\overline{u} - u)^2$, i.e., $\int (\overline{u} - u) \cdot \mu(u)\, du$.
- To minimize this expression, we differentiate it relative to $\overline{u}$ and equate the result to 0; thus

$$\overline{u} = \frac{\int u \cdot \mu(u)\, du}{\int \mu(u)\, du}.$$

- This is known as *centroid defuzzification.*
- The resulting *fuzzy control* has indeed been very successful in many applications, from rice cookers to trains.

# 27. Fuzzy Control: Summarizing

- We start with the if-then expert rules of the type "If $x$ is small, and ..., then $y$ is small":

$$A_{r1}(x_1)\,\&\,\ldots\,\&\,A_{rn}(x_n)\Rightarrow B_r(y).$$

- Here $x_1,\ldots,x_n$ are inputs, and $A_{ri}$ and $B_r$ are words that describe properties of inputs and output.
- For each words $w$ used in these rules, we pick several values $x^{(1)},\ldots,x^{(k)}$.
- We determine the degrees of confidence $\mu_w(x^{(1)}),\ldots,\mu_w(x^{(n)})$ that these values satisfy the property $w$.
- Then, we use some interpolation technique to determine the membership functions $\mu_w(x)$ for all $x$.
- We choose "and"- and "or"-operations $f_\&(a,b)$ and $f_\vee(a,b)$.

## 28. Fuzzy Control: Summarizing (cont-d)

- For each rule $r$, and for each possible values of input and output, we compute the *firing degrees*

$$d_r(x_1, \ldots, x_n, y) = f_\&(\mu_{r1}(x_1), \ldots, \mu_{rn}(x_n), \mu_r(y)).$$

- Then we compute the membership function for control

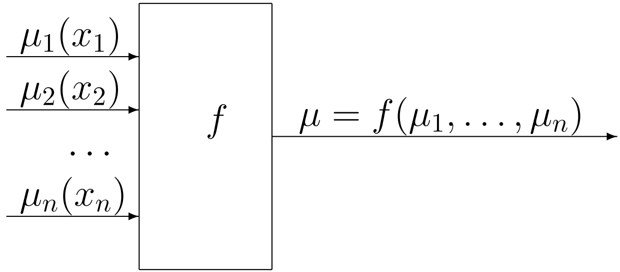
$$\mu_C(x_1, \ldots, x_n, y) = f_\vee(d_1(x_1, \ldots, x_n, y), \ldots, d_R(x_1, \ldots, x_n, y)).$$

- For every input $x_1, \ldots, x_n$, $\mu_C(y)$ is the degree of confidence that $y$ is a reasonable control.

- If needed, we can then get a single recommended control value $\bar{y}$:

$$\bar{y} = \frac{\int y \cdot \mu_C(y)\,dy}{\int \mu_C(y)\,dy}.$$

# 29. Fuzzy Computations: A Problem




- *Given:* an algorithm $y = f(x_1, \dots, x_n)$ and $n$ fuzzy numbers $\mu_i(x_i)$.
- *Compute:* $\mu(y) = \max\limits_{x_1,\dots,x_n : f(x_1,\dots,x_n)=y} \min(\mu_1(x_1), \dots, \mu_n(x_n))$.
- *Motivation:* $y$ is a possible value of $Y \leftrightarrow \exists x_1, \dots, x_n$ s.t. each $x_i$ is a possible value of $X_i$ and $f(x_1, \dots, x_n) = y$.
- *Details:* "and" is min, $\exists$ ("or") is max, hence

$$\mu(y) = \max_{x_1,\dots,x_n} \min(\mu_1(x_1), \dots, \mu_n(x_n), t(f(x_1, \dots, x_n) = y)).$$

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## 30. Fuzzy Computations: Reduction to Interval Computations

- *Problem (reminder):*
  - *Given:* an algorithm $y = f(x_1, \ldots, x_n)$ and $n$ fuzzy numbers $X_i$ described by membership functions $\mu_i(x_i)$.
  - *Compute:* $Y = f(X_1, \ldots, X_n)$, where $Y$ is defined by Zadeh's extension principle:

$$\mu(y) = \max_{x_1,\ldots,x_n : f(x_1,\ldots,x_n)=y} \min(\mu_1(x_1), \ldots, \mu_n(x_n)).$$

- *Idea:* represent each $X_i$ by its $\alpha$-cuts

$$X_i(\alpha) = \{x_i : \mu_i(x_i) \geq \alpha\}.$$

- *Advantage:* for continuous $f$, for every $\alpha$, we have

$$Y(\alpha) = f(X_1(\alpha), \ldots, X_n(\alpha)).$$

- *Resulting algorithm:* for $\alpha = 0, 0.1, 0.2, \ldots, 1$ apply interval computations techniques to compute $Y(\alpha)$.