Students Who Took the Class Help Students Who Are Taking It: What Is the Best Arrangement?

Olga Kosheleva[1] and Vladik Kreinovich[2]
[1]Department of Teacher Education
[2]Department of Computer Science
University of Texas at El Paso
500 W. University, El Paso, Texas 79968, USA
olgak@utep.edu, vladik@utep.edu

Students need help
Students helping students
This benefits helpers too
Numerical description . . .
Numerical description . . .
How effective is help?
Resulting formulation . . .
Resulting solution
More accurate models . . .

1. Students need help

- Studying at a university if not easy.
- Many initially promising students do not succeed.
- It is therefore necessary to provide as much help as possible to the students.
- Helping students is one of the main obligations of instructors and of their Teaching Assistants.
- However, the available time of instructors and Teaching Assistants is limited.
- So, especially for big classes, students need additional help.

Students need help
Students helping students
This benefits helpers too
Numerical description . . .
Numerical description . . .
How effective is help?
Resulting formulation . . .
Resulting solution
More accurate models . . .

2. Students helping students

- Who can provide such help?
- Naturally, only people who already know this material can meaningfully help:
  - in addition to instructors and Teaching Assistants,
  - there is another group of folks who know this material – and who can thus help:
  - students who have already taken this class.
- So, a natural idea is:
  - to have these students help students who are currently taking this class,
  - i.e., help students who are currently struggling with this class's material.

Students need help
Students helping students
This benefits helpers too
Numerical description . . .
Numerical description . . .
How effective is help?
Resulting formulation . . .
Resulting solution
More accurate models . . .

3. This benefits helpers too

- Such help clearly benefits students who are currently taking this class.
- But why should other students help them?
- Some universities can afford to pay student helpers, but such ability to pay is rare.
- Good news is that such arrangement helps helpers as well.
- Every instructor knows that the best way to learn the material is to teach it:
  - by repeating each part of the material many times,
  - by answering numerous questions,
  - the instructor gets a better and better mastery of the material.

Students need help
Students helping students
This benefits helpers too
Numerical description . . .
Numerical description . . .
How effective is help?
Resulting formulation . . .
Resulting solution
More accurate models . . .

4. This benefits helpers too (cont-d)

- Similarly, in the process of helping other students, student helpers get a better mastery of the material.
- Because helping is beneficial to both groups of students:
  - at some universities – e.g., at the University of California at Berkeley,
  - it is, in effect, institutionalized.
- Students are supposed
  - to spend some time during their next class
  - on helping students who are currently taking the previous class.

Students need help
Students helping students
This benefits helpers too
Numerical description . . .
Numerical description . . .
How effective is help?
Resulting formulation . . .
Resulting solution
More accurate models . . .

5. A natural question

- A natural question is: what is the best way to pair the students?
- Who should help whom?
- This is the question that we will answer in this paper.
- To answer the above question, let us reformulate this question in precise terms.

Students need help
Students helping students
This benefits helpers too
Numerical description . . .
Numerical description . . .
How effective is help?
Resulting formulation . . .
Resulting solution
More accurate models . . .

6. Numerical description of helpers

- The better was the helping student's performance in the class, the more this student can help others.
- Let us denote the helping ability of the i-th helping student by h_i.
- As a measure of such an ability, we can take, e.g., the student's grade in the previous class.
- To be more precise, the 0-to-100 number of points that led to this specific grade.
- For simplicity, we can normalize this numerical characteristic – by dividing it by its largest possible value.
- This way, the corresponding numerical characteristic takes values from 0 to 1.

Students need help
Students helping students
This benefits helpers too
Numerical description . . .
Numerical description . . .
How effective is help?
Resulting formulation . . .
Resulting solution
More accurate models . . .

7. Numerical description of helpees

- Students who currently take this class also have different success rates.
- Let us denote the success level of the i-th helpee student by s_i.
- Again, as a numerical measure of this success, we can take:
  - the student's grade-so-far,
  - or the student's average grade in the previous classes.
- This grade can also be normalized, so that it takes values from 0 to 1.

Students need help
Students helping students
This benefits helpers too
Numerical description . . .
Numerical description . . .
How effective is help?
Resulting formulation . . .
Resulting solution
More accurate models . . .

8. Numerical description of helpees (cont-d)

- The smaller the value s_i, the larger the student's need for help.
- We can estimate this need for help n_i as the difference between the ideal value 1 and the current value s_i:

$$n_i = 1 - s_i.$$

Students need help
Students helping students
This benefits helpers too
Numerical description . . .
Numerical description . . .
How effective is help?
Resulting formulation . . .
Resulting solution
More accurate models . . .

9. How effective is help?

- The effectiveness of help cannot exceed the ability of the helping student:
  - if a student is not very strong in this material,
  - this student is not of much help.
- Similarly:
  - if a helpee already knows everything – or at least his knowledge gap is small,
  - then the help cannot exceed this gap.
- On the other hand:
  - if the helpee's gap is exactly equal to the knowledge that the helper has, and
  - both the helper and the helpee work hard,
  - they get the maximum effect: all gaps are covered.

Students need help
Students helping students
This benefits helpers too
Numerical description . . .
Numerical description . . .
How effective is help?
Resulting formulation . . .
Resulting solution
More accurate models . . .

10. How effective is help (cont-d)

- So, in general, the efficiency of a helper-helpee arrangement:
  - when the helper i is paired with the helpee $p(i)$,
  - is proportional to the smallest of the two values, i.e., to: $\min(h_i, n_{p(i)})$.
- We want to maximize the overall effectiveness, i.e., the sum of the effectivenesses of all the helper-helpee pairs.
- Thus, we arrive at the following precise formulation of the problem.

Students need help
Students helping students
This benefits helpers too
Numerical description . . .
Numerical description . . .
How effective is help?
Resulting formulation . . .
Resulting solution
More accurate models . . .

11. Resulting formulation of the mathematical problem

- We have s students potentially needing help, and we have s students who can potentially deliver this help.
- We know how much help h_i each helper can provide.
- We also know how much help n_i each helpee needs.
- We need to find a permutation

$$p : \{1, \ldots, s\} \to \{1, \ldots, s\}.$$

- We want to find the permutation for which the following sum is the largest possible:

$$\min(h_1, n_{p(1)}) + \ldots + \min(h_s, n_{p(s)}).$$

Students need help
Students helping students
This benefits helpers too
Numerical description . . .
Numerical description . . .
How effective is help?
Resulting formulation . . .
Resulting solution
More accurate models . . .

12. Resulting solution

- Let us prove that the largest possible value of the above objective function corresponds to the case when:
  - the best performing helper is matched to the worst performing helpee,
  - the second best helper is matched to the second worst helpee, etc.
- In other words:
  - we sort helpers in the increasing order of their helping ability h_i, and
  - we sort helpees in the increasing order of their helping need n_i.

Students need help
Students helping students
This benefits helpers too
Numerical description . . .
Numerical description . . .
How effective is help?
Resulting formulation . . .
Resulting solution
More accurate models . . .

13. Resulting solution (cont-d)

- Then:
  - the student with the smallest possible value h_1 of the helping ability
  - is matched with the student with the smallest possible value n_1 of the helping need;
  - the student with the second smallest value h_2 of the helping ability
  - is matched with the student with the second smallest possible value n_2 of the helping need, etc.
- Finally:
  - the student with the largest possible value h_s of the helping ability
  - is matched with the student with the largest possible value n_s of the helping need.

Students need help
Students helping students
This benefits helpers too
Numerical description . . .
Numerical description . . .
How effective is help?
Resulting formulation . . .
Resulting solution
More accurate models . . .

14. How to prove that this arrangement is optimal

- In the optimal arrangement, if $h_i < h_j$, then we should have $n_{p(i)} \le n_{p(j)}$.
- To prove that such an arrangement is optimal, it is sufficient to prove that:
  - if for two pairs $(i, p(i))$ and $(j, p(j))$, we have an opposite relation, i.e., we have

$$h_i < h_j \text{ and } n_{p(i)} > n_{p(j)},$$

  - then, by swapping $p(i)$ and $p(j)$, we will increase (or at least not decrease) the objective function.

15. How to prove optimality (cont-d)

- So, whatever arrangement we start with:
  - by applying this swap again and again,
  - we will arrive at the arrangement in which the orders of h_i and $n_{p(i)}$ are the same,
  - and the value of the objective function will only increase (or remain the same).
- Thus, the largest possible value of the objective function is indeed attained when the orders coincide.
- We want to prove that when $h_i < h_j$ and $n_{p(i)} > n_{p(j)}$, then $v \le V$, where we denoted

$$v = \min(h_i, n_{p(i)}) + \min(h_j, n_{p(j)}) \text{ and}$$

$$V = \min(h_i, n_{p(j)}) + \min(h_j, n_{p(i)}).$$

- Let us consider all possible locations of the values $n_{p(j)} < n_{p(i)}$ with respect to h_i and h_j.

16. Case 1

- First, let us consider the cases when $n_{p(j)} \le h_i < h_j$.
- In this case, the value $n_{p(i)}$ can be:

  (1) either before h_i,

  (2) or between h_i and h_j,

  (3) or after h_j.
- Let us consider these three subcases one by one.
- Subcase 1.1. In this subcase, $n_{p(j)} < n_{p(i)} \le h_i < h_j$.
- Here, $v = n_{p(i)} + n_{p(j)}$ and $V = n_{p(i)} + n_{p(j)}$, so $v = V$: the decided inequality is satisfied.
- Subcase 1.2. In this subcase, $n_{p(j)} \le h_i < n_{p(i)} \le h_j$.
- Here, $v = h_i + n_{p(j)}$ and $V = n_{p(j)} + n_{p(i)}$.
- Since $h_i < n_{p(i)}$, we have $v < V$.

17. Case 1 (cont-d)

- Subcase 1.3. In this subcase, $n_{p(j)} \leq h_i < h_j \leq n_{p(i)}$.
- Here, $v = h_i + n_{p(j)}$ and $V = n_{p(j)} + h_j$.
- Since $h_i < h_j$, we also have $v < V$.

Students need help
Students helping students
This benefits helpers too
Numerical description . . .
Numerical description . . .
How effective is help?
Resulting formulation . . .
Resulting solution
More accurate models . . .

18. Case 2

- Let us now consider the case when $h_i \le n_{p(j)} < h_j$.
- In this case, the value $n_{p(i)}$ can be:

  (1) either between h_i and h_j,

  (2) or after h_j.
- Let us consider these two subcases one by one.
- Subcase 2.1. In this subcase, $h_i \le n_{p(j)} < n_{p(i)} \le h_j$.
- Here, $v = h_i + n_{p(j)}$ and $V = h_i + n_{p(i)}$.
- Since $n_{p(j)} < n_{p(i)}$, we have $v < V$.
- Subcase 2.2. In this subcase, $h_i \le n_{p(j)} < h_j \le n_{p(i)}$.
- Here, $v = h_i + n_{p(j)}$ and $V = h_i + h_j$.
- Since $n_{p(j)} < h_j$, we have $v < V$.

Students need help
Students helping students
This benefits helpers too
Numerical description . . .
Numerical description . . .
How effective is help?
Resulting formulation . . .
Resulting solution
More accurate models . . .

19. Case 3

- Finally, we need to consider the case when $h_j \le n_{p(j)}$.
- In this case, there is only one possible arrangement:

$$h_i < h_j \le n_{p(j)} < n_{p(i)}.$$

- Here, $v = h_i + h_j$ and $V = h_i + h_j$.
- So, $v = V$.

Students need help
Students helping students
This benefits helpers too
Numerical description . . .
Numerical description . . .
How effective is help?
Resulting formulation . . .
Resulting solution
More accurate models . . .

20. Summarizing

- In all possible cases, we have $v \leq V$.
- Thus, indeed, in all the cases, swapping either increases the value of the objective function or leaves it intact.
- Thus, we have proved that:
  - the above arrangement – in which the orders between h_i and n_i coincide,
  - is indeed optimal.
- In other words, we arrive at the following conclusion.

Students need help
Students helping students
This benefits helpers too
Numerical description . . .
Numerical description . . .
How effective is help?
Resulting formulation . . .
Resulting solution
More accurate models . . .

21. Conclusion

- The optimal pairing between helpers and helpees is as follows.
- We sort all potential helpers:
  - in the increasing order of their ability to help,
  - i.e., in the increasing order of the grade that they got when they took this class:

$$h_1 \le h_2 \le \ldots \le h_s.$$

- Then, we sort all potential helpees:
  - in the increasing order of their need for help,
  - i.e., equivalently, in the decreasing order of the grade-so-far:

$$n_1 \le n_2 \le \ldots \le n_s.$$

Students need help
Students helping students
This benefits helpers too
Numerical description . . .
Numerical description . . .
How effective is help?
Resulting formulation . . .
Resulting solution
More accurate models . . .

22. Conclusion (cont-d)

- Then, we match:
  - the 1st helper with the 1st helpee,
  - the 2nd helper with the 2nd helpee, . . . , and
  - the s-th helper with the s-th helpee.

Students need help
Students helping students
This benefits helpers too
Numerical description . . .
Numerical description . . .
How effective is help?
Resulting formulation . . .
Resulting solution
More accurate models . . .

23. More accurate models are possible

- In the above arguments:
  - we used a simplified model of the helping efficiency,
  - according to which the efficiency of each helping-helper pair is equal to $\min(h_i, n_{p(i)})$.
- This seems like a good first approximation model.
- However, a natural question is: what if we use a more accurate model?
- Will the recommendation change?

Students need help
Students helping students
This benefits helpers too
Numerical description . . .
Numerical description . . .
How effective is help?
Resulting formulation . . .
Resulting solution
More accurate models . . .

24. More accurate models are possible (cont-d)

- One possible alternative model is to estimate the helping efficiency:
  - not as the minimum,
  - but as the product of the values h_i and $n_{p(i)}$, i.e., as $h_i \cdot n_{p(i)}$.
- We can use even more sophisticated models, in which the efficiency of a helper-helpee pair is described as

$$f(h_i, n_{p(i)}) \text{ for some function } f(a, b).$$

Students need help
Students helping students
This benefits helpers too
Numerical description . . .
Numerical description . . .
How effective is help?
Resulting formulation . . .
Resulting solution
More accurate models . . .

25. Analysis of the problem

- For which functions $f(a, b)$ is the above argument applicable?
- It is applicable if for all $a < A$ and $b < B$, we have

$$f(a, b) + f(A, B) \geq f(a, B) + f(A, b).$$

- This inequality can be reformulated in the equivalent form $f(a, B) - f(a, b) \leq f(A, B) - f(A, b)$.
- If we take $B = b + h$, divide both sides by h, and take a limit $h \to 0$.
- Then we get an inequality between the values of the partial derivative with respect to b:

$$\frac{\partial f}{\partial b}(a, b) \leq \frac{\partial f}{\partial b}(A, b).$$

26. Analysis of the problem (cont-d)

- Vice versa:
  - if this inequality between the values of the partial derivative holds for all a, A, and b,
  - then by integrating over b, we can get the desired inequality between the differences.
- The inequality between the values of the partial derivative simply means that:
  - the partial derivative
  - is an increasing function of the first argument a.
- This is equivalent to the derivative of this expression with respect to a being non-negative, i.e., that:

$$\frac{\partial^2 f}{\partial a \partial b} \geq 0 \text{ for all } a \text{ and } b.$$

Students need help
Students helping students
This benefits helpers too
Numerical description . . .
Numerical description . . .
How effective is help?
Resulting formulation . . .
Resulting solution
More accurate models . . .

27. Analysis and the resulting conclusion

- One can easily check that this inequality holds:
  - for the original function $f(a, b) = \min(a, b)$,
  - for the product function $f(a, b) = a \cdot b$ (for which the above double derivative is always equal to 1),
  - and for many other models.
- Thus, we can arrive at the following conclusion.
- The optimality of the above arrangement:
  - holds not only for the simplest model $\min(h_i, n_{p(i)})$ of helping efficiency,
  - it holds for any model $f(h_i, n_{p(i)})$ for which

$$\frac{\partial^2 f}{\partial a \partial b} \geq 0,$$

  - in particular, for the product model $f(a, b) = a \cdot b$.

28. Acknowledgments

This work was supported in part by the National Science Foundation grants:

- 1623190 (A Model of Change for Preparing a New Generation for Professional Practice in Computer Science);
- HRD-1834620 and HRD-2034030 (CAHSI Includes).

It was also supported by the program of the development of the Scientific-Educational Mathematical Center of Volga Federal District No. 075-02-2020-1478.

Students need help
Students helping students
This benefits helpers too
Numerical description . . .
Numerical description . . .
How effective is help?
Resulting formulation . . .
Resulting solution
More accurate models . . .